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AUTHOR Herriot, Sarah T.; And Others
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ABSTRACT

This teacher's guide is designed for use with the SMSG textbook "Calculus of Elementary Functions." It contains solutions to exercises found in Chapter 9, Integration Theory and Technique; Chapter 10, Simple Differential Equations; Appendix 5, Area and Integral; Appendix 6; Appendix 7, Continuity Theory; and Appendix 8, More About Integrals. (MK)

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**SCHOOL
MATHEMATICS
STUDY GROUP**

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CALCULUS OF ELEMENTARY FUNCTIONS

Part IV

Teacher's Commentary

(Preliminary Edition)

SE 030 450

CALCULUS OF ELEMENTARY FUNCTIONS

Part IV

Teacher's Commentary

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The following is a list of all those who participated in the preparation of this volume:

Sarah T. Herriot, Gunn High School, Palo Alto, California
Desmond T. Jenkins, Palo Alto Senior High School, Palo Alto
C. W. Leeds, III, Berkshire School, Sheffield, Mass.
Paul C. Shields, Wayne State University, Detroit

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Chapter 9

INTEGRATION THEORY AND TECHNIQUE

Solutions Exercises 9-1

1. (a) $\int \frac{x^2}{x^3 + a^3}\,dx; \qquad u = x^3 + a^3$

$$du = 3x^2\,dx$$

$$\frac{1}{3}\,du = x^2\,dx$$

$$\frac{1}{3}\int u^{-1}du = \frac{1}{3}\log_e u$$

$$= \frac{1}{3}\log_e(x^3 + a^3)$$

(b) $\int x^3\sqrt[7]{1 - x^4}\,dx; \qquad u = 1 - x^4$

$$du = -4x^3\,dx$$

$$-\frac{1}{4}\,du = x^3\,dx$$

$$-\frac{1}{4}\int u^{1/7}du = -\frac{1}{4}\,\frac{u^{8/7}}{\frac{8}{7}}$$

$$= \frac{-7}{32}(1 - x^4)^{8/7}$$

(c) $\int \frac{(a + b\sqrt{x})^{13}}{\sqrt{x}}\,dx,\ b \neq 0; \qquad u = a + b\sqrt{x}$

$$du = \frac{b}{2\sqrt{x}}\,dx$$

$$\frac{2}{b}\,du = \frac{dx}{\sqrt{x}}$$

$$\frac{2}{b}\int u^{13}\,du = \frac{1}{7b}(a + b\sqrt{x})^{14}$$

(d) $\int \frac{x^2 + 1}{x - 1} dx;$ $\quad u = x - 1$

$$du = dx$$
$$x = u + 1$$
$$x^2 = u^2 + 2u + 1$$

$$\int \frac{(u^2 + 2u + 1) + 1}{u} du = \int (u + 2 + \frac{2}{u}) du$$
$$= \frac{1}{2}(x^2 + 2x - 3 + 4 \log_e (x - 1))$$

(e) $\int \frac{x}{x^2 + a^2} dx;$ $\quad u = x^2 + a^2$

$$du = 2x\ dx$$
$$\frac{1}{2} du = x\ dx$$

$$\frac{1}{2} \int u^{-1} du = \frac{1}{2} \log_e (x^2 + a^2) = \log_e \sqrt{x^2 + a^2}$$

(f) $\int \frac{x}{x^4 + a^2} dx,\ a \neq 0;$ $\quad u = \frac{x^2}{a}$ and $x^2 = au$

$$du = \frac{2}{a} x\ dx$$
$$\frac{a}{2} du = x\ dx$$

$$\frac{a}{2} \int \frac{1}{a^2 u^2 + a^2} du = \frac{a}{2} \cdot \frac{1}{a^2} \int \frac{1}{1 + u^2} du$$
$$= \frac{1}{2a} \arctan \frac{x^2}{a}$$

(g) $\int (\cos x) e^{\sin x} dx;$ $\quad u = \sin x$

$$du = \cos x\ dx$$

$$\int e^u du = e^{\sin x} \cos x\ dx$$

(h) $\int \frac{ae^x}{b + ce^x} dx,\ c \neq 0;$ $\quad u = b + ce^x$

$$du = ce^x\ dx$$
$$\frac{1}{c} du = e^x\ dx$$

$$\frac{a}{c} \int u^{-1} du = \frac{a}{c} \log_e (b + ce^x)$$

(i) $\int \sec x \, dx; \quad u = \sec x + \tan x$

$$du = (\sec x \tan x + \sec^2 x)dx$$

$$du = \sec x(\tan x + \sec x)dx$$

$$\frac{du}{u} = \sec x \, dx$$

$$\int \frac{du}{u} = \log_e (\sec x + \tan x)$$

2. (a) $\int e^{2x} \, dx; \quad u = 2x$

$$\frac{1}{2} du = dx$$

$$\frac{1}{2} \int e^u \, du = \frac{1}{2} e^{2x}$$

(b) $\int (1 - \frac{1}{2} x)^{10} \, dx; \quad u = 1 - \frac{1}{2} x$

$$du = - \frac{1}{2} dx$$

$$-2du = dx$$

$$-2 \int u^{10} \, du = - \frac{2}{11}(1 - \frac{1}{2} x)^{11}$$

(c) $\int \sin ax \, dx; \quad u = ax$

$$du = a \, dx$$

$$\frac{1}{a} du = dx$$

$$\frac{1}{a} \int \sin u \, du = - \frac{1}{a} \cos ax$$

(d) $\int \sqrt[4]{3x + 1} \, dx; \quad u = 3x + 1$

$$\frac{1}{3} du = dx$$

$$\frac{1}{3} \int u^{1/4} \, du = \frac{4}{15} (3x + 1)^{5/4}$$

(e) $\int \frac{1}{2 - 3x}\,dx;\qquad u = 2 - 3x$

$$-\frac{1}{3}\,du = dx$$

$$-\frac{1}{3}\int u^{-1}du = -\frac{1}{3}\log_e(2 - 3x)$$

(f) $\int \frac{1}{\sqrt{(1 - 5x)^3}}\,dx;\qquad u = 1 - 5x$

$$-\frac{1}{5}\,du = dx$$

$$-\frac{1}{5}\int u^{-3/2}du = \frac{2}{5\sqrt{1 - 5x}}$$

(g) $\int \frac{1}{a^2 + x^2}\,dx;\qquad u = \frac{x}{a}$ then $au = x$

$$adu = dx$$

$$a\int \frac{1}{a^2 + a^2u^2}\,du = \frac{1}{a}\int \frac{1}{1 + u^2}\,du$$

$$= \frac{1}{a}\arctan\frac{x}{a}$$

(h) $\int \tan(\frac{1}{2}x - 3)dx;\qquad u = \frac{1}{2}x - 3$

$$2du = dx$$

$$2\int \tan u\,du = -2\log_e\cos(\frac{1}{2}x - 3)$$

3. (a) $\int (4 - 3x^2)^6\,x\,dx;\qquad u = 4 - 3x^2$

$$du = -6x\,dx$$

$$-\frac{1}{6}\,du = x\,dx$$

$$-\frac{1}{6}\int u^6\,du = -\frac{1}{42}(4 - 3x^2)^7 = \frac{1}{42}(3x^2 - 4)^7$$

(b) $\int \cos^5 x \sin x \, dx;$ $u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$- \int u^5 \, du = - \frac{1}{6} \cos^6 x$

(c) $\int \sin^2 2x \cos 2x \, dx;$ $u = \sin 2x$

$du = 2 \cos 2x \, dx$

$\frac{1}{2} du = \cos 2x \, dx$

$\frac{1}{2} \int u^2 \, du = \frac{1}{6} \sin^3 2x$

(d) $\int \frac{e^{1/x}}{x^2} \, dx;$ $u = \frac{1}{x}$

$du = - \frac{1}{x^2} \, dx$

$-du = \frac{1}{x^2} \, dx$

$- \int e^u \, du = -e^{1/x}$

(e) $\int x\sqrt{1 + 4x^2} \, dx;$ $u = 1 + 4x^2$

$du = 8x \, dx$

$\frac{1}{8} du = x \, dx$

$\frac{1}{8} \int u^{1/2} \, du = \frac{1}{12}(1 + 4x^2)^{3/2}$

(f) $\int \frac{(\log_e x)^2}{x} \, dx;$ $u = \log_e x$

$du = \frac{1}{x} \, dx$

$\int u^2 \, du = \frac{1}{3}(\log_e x)^3$

(g) $\int \frac{\cos \sqrt{2x}}{\sqrt{x}}\,dx;$ $\quad u = \sqrt{2x}$

$$du = \frac{\sqrt{2}}{2\sqrt{x}}\,dx$$

$$\sqrt{2}\,du = \frac{dx}{\sqrt{x}}$$

$$\sqrt{2}\int \cos u\,du = \sqrt{2}\sin\sqrt{2x}$$

(h) $\int \frac{\sin x}{(a + b\cos x)^2}\,dx;$ $\quad u = a + b\cos x$

$$du = -b\sin x$$

$$-\frac{1}{b}\,du = \sin x\,dx$$

$$-\frac{1}{b}\int u^{-2}\,du = \frac{1}{b(a + b\cos x)}$$

(i) $\int \frac{x^2}{(4x^3 - 1)^{3/2}}\,dx;$ $\quad u = 4x^3 - 1$

$$du = 12x^2\,dx$$

$$\frac{1}{12}\,du = x^2\,dx$$

$$\frac{1}{12}\int u^{-3/2}\,du = \frac{-1}{6(4x^3 - 1)^{1/2}}$$

(j) $\frac{x}{1 + x^2}\,dx;$ $\quad u = 1 + x^2$

$$du = 2x\,dx$$

$$\frac{1}{2}\,du = x\,dx$$

$$\frac{1}{2}\int u^{-1}\,du = \frac{1}{2}\log_e(1 + x^2) = \log_e\sqrt{1 + x^2}$$

(k) $\int \frac{x}{1 + x^4}\,dx;$ $\qquad u = x^2$

$du = 2x\,dx$

$\frac{1}{2}\,du = x\,dx$

$$\frac{1}{2}\int \frac{1}{1 + u^2}\,du = \frac{1}{2}\arctan x^2$$

(ℓ) $\int \frac{x}{\sqrt{1 - 9x^4}}\,dx;$ $\qquad u = 3x^2$

$du = 6x\,dx$

$\frac{1}{6}\,du = x\,dx$

$$\frac{1}{6}\int \frac{1}{\sqrt{1 - u^2}}\,du = \frac{1}{6}\,\text{arc sin}\;3x^2$$

(m) $\int \sin^2 x \cos^3 x\,dx$: Let $\cos^3 x = \cos x(1 - \sin^2 x)$

$\int \sin^2 x(1 - \sin^2 x)\cos x\,dx = \int(\sin^2 x - \sin^4 x)\cos x\,dx;$

$u = \sin x$

$du = \cos x\,dx$

$$\int(u^2 - u^4)\,du = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x$$

(n) $\int \sin^3 4x \cos^8 4x\,dx;$ $\qquad \sin^3 4x = \sin 4x(1 - \cos^2 4x)$

$\int(1 - \cos^2 4x)\cos^8 4x \sin 4x\,dx$

$\int(\cos^8 4x - \cos^{10} 4x)\sin 4x\,dx;$ $\qquad u = \cos 4x$

$du = -4\sin 4x\,dx$

$-\frac{1}{4}\,du = \sin 4x\,dx$

$$-\frac{1}{4}\int(u^8 - u^{10})\,du = -\frac{1}{36}\cos^9 4x + \frac{1}{44}\cos^{11} 4x$$

4. (a) $\int_2^3 \frac{1}{(2x+1)^2}\,dx; \qquad u = 2x + 1$

$$du = 2\,dx$$

$$\frac{1}{2}\,du = dx$$

$$\frac{1}{2}\int_5^7 u^{-2}\,du = -\frac{1}{2}u^{-1}\Big|_5^7$$

$$= -\frac{1}{2}[\frac{1}{7} - \frac{1}{5}]$$

$$= \frac{-(5-7)}{70} = \frac{1}{35}$$

(b) $\int_0^{\pi} \cos^4 x \sin x\,dx; \qquad u = \cos x$

$$du = -\sin x\,dx$$

$$-du = \sin x\,dx$$

$$-\int_1^{-1} u^4\,du = -\frac{1}{5}u^5\Big|_1^{-1}$$

$$= -\frac{1}{5}[-1 - (+1)] = \frac{2}{5}$$

(c) $\int_0^{\pi/3} \cos 4x\,dx; \qquad u = 4x$

$$\frac{1}{4}\,du = dx$$

$$\frac{1}{4}\int_0^{4\pi/3} \cos u\,du = \frac{1}{4}\sin u\Big|_0^{4\pi/3}$$

$$= \frac{1}{4}[-\frac{\sqrt{3}}{2} - 0] = -\frac{\sqrt{3}}{8}$$

(d) $\int_{-1/2}^{0} (2x+1)^{17}dx; \qquad u = 2x + 1$

$$\frac{1}{2}\,du = dx$$

$$\frac{1}{2}\int_0^1 u^{17}\,du = \frac{1}{36}u^{18}\Big|_0^1$$

$$= \frac{1}{36}$$

(e) $\int_{-1}^{0} \sqrt{1 + x}\, dx;$ $\qquad u = 1 + x$

$du = dx$

$$\int_{2}^{1} u^{1/2}\, du = \frac{2}{3} u^{3/2} \Big|_{2}^{1}$$

$$= \frac{2}{3}[1 - 2\sqrt{2}]$$

(f) $\int_{-1}^{1} x \sqrt{1 - x^2}\, dx;$ $\qquad u = 1 - x^2$

$du = -2x\, dx$

$-\frac{1}{2}\, du = x\, dx$

$$-\frac{1}{2} \int_{0}^{0} u^{1/2}\, du = 0$$

(g) $\int_{0}^{1/2} \frac{1}{1 + 4x^2}\, dx;$ $\qquad u = 2x$

$\frac{1}{2}\, du = dx$

$$\frac{1}{2} \int_{0}^{1} \frac{1}{1 + u^2}\, du = \frac{1}{2} \arctan u \Big|_{0}^{1}$$

$$= \frac{1}{2}[\frac{\pi}{4} - 0]$$

$$= \frac{\pi}{8}$$

(h) $\int_{-1/2}^{1/2} \frac{1}{\sqrt{9 - x^2}}\, dx;$ $\qquad u = \frac{x}{3}$ thus $3u = x$

$3\, du = dx$

$$3 \int_{-1/6}^{1/6} \frac{1}{\sqrt{9 - 9u^2}}\, du = \frac{3}{3} \int_{-1/6}^{1/6} \frac{1}{\sqrt{1 - u^2}}\, du$$

$$= \text{arc sin } u \Big|_{-1/6}^{1/6}$$

(arc sin $.16\overline{66}$ = .177 from Table 3.)

$$\approx .177 - (-.177)$$

$$\approx .354$$

(i) $\int_1^2 \frac{\log_e x}{x}\,dx;$ $\qquad u = \log_e x$

$$du = \frac{1}{x}\,dx$$

$$\int_{\log_e 1}^{\log_e 2} u\,du = \frac{1}{2}u^2 \Big|_0^{\log_e 2}$$

$$= \frac{1}{2}(\log_e 2)^2 \approx .24$$

(j) $\int_0^1 x^3 \sqrt{1 - x^2}\,dx;$ $\qquad u = 1 - x^2, \quad x^2 = 1 - u$

$$du = -2x\,dx$$

$$-\frac{1}{2}\,du = x\,dx$$

$$-\frac{1}{2}\int_1^0 (1 - u)u^{1/2}\,du = -\frac{1}{2}\int_1^0 (u^{1/2} - u^{3/2})du$$

$$= -\frac{1}{2}\left(\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right)\Big|_1^0$$

$$= -\frac{1}{2}\left[0 - \left(\frac{2}{3} - \frac{2}{5}\right)\right] = \frac{2}{15}$$

(k) $\int_{-1}^1 x^2 e^{x^3}\,dx;$ $\qquad u = x^3$

$$du = 3x^2\,dx$$

$$\frac{1}{3}\,du = x^2\,dx$$

$$\frac{1}{3}\int_{-1}^1 e^u\,du = \frac{1}{3}e^u \Big|_{-1}^1$$

$$= \frac{1}{3}\left[e - \frac{1}{e}\right]$$

(*l*) $\int_0^{\sqrt{\pi/2}} x \sin(2x^2)dx;$ $\qquad u = 2x^2$

$$du = 4x\,dx$$

$$\frac{1}{4}\,du = x\,dx$$

$$\frac{1}{4}\int_0^{\pi} \sin u\,du = -\frac{1}{4}\cos u \Big|_0^{\pi} = -\frac{1}{4}[-1 - (+1)]$$

$$= \frac{1}{2}$$

5. (a) $\int \frac{\cos x}{\sqrt{9 - \sin^2 x}}\, dx;$ $\quad u = \sin x$

$du = \cos x\, dx$

$\int \frac{du}{\sqrt{9 - u^2}}\ ;$ $\quad v = \frac{u}{3}$ or $3v = u$

$3dv = du$

$$3\int \frac{dv}{\sqrt{9 - 9v^2}} = 3\int \frac{dv}{3\sqrt{1 - v^2}}$$

$$= \frac{3}{3} \text{arc sin } v$$

$$= \text{arc sin } \frac{u}{3}$$

$$= \text{arc sin } (\frac{1}{3} \sin x)$$

(b) $\int \frac{x^2}{2 + x^6}\, dx;$ $\quad u = x^3$

$du = 3x^2\, dx$

$\frac{1}{3}\, du = x^2\, dx$

$\frac{1}{3} \int \frac{1}{2 + u^2}\, du;$ $\quad v = \frac{u}{\sqrt{2}}$

$\sqrt{2}\, v = u$

$\sqrt{2}\, dv = du$

$$\frac{\sqrt{2}}{3} \int \frac{1}{2(1 + v^2)}\, dv = \frac{\sqrt{2}}{6} \text{arc tan } v$$

$$= \frac{\sqrt{2}}{6} \text{arc tan } \frac{u}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{6} \text{arc tan } \frac{x^3}{\sqrt{2}}$$

(c) $\int \frac{1}{\sqrt{x} + x}\,dx;$ $\qquad u = \sqrt{x}$

$$du = \frac{1}{2\sqrt{x}}\,dx$$

$$2u\,du = dx$$

$$2\int \frac{u\,du}{u + u^2} = 2\int \frac{du}{1 + u}; \qquad v = 1 + u$$

$$dv = du$$

$$2\int \frac{1}{v}\,dv = 2\log_e v$$

$$= 2\log_e (1 + u)$$

$$= 2\log_e (1 + \sqrt{x})$$

6. (a) $\int x^2 \sin (x - 1)dx;$ $\qquad u = (x - 1)$

$$u + 1 = x$$

$$du = dx$$

$$\int (u^2 + 2u + 1)\sin u\,du = \int u^2 \sin u\,du + 2\int u \sin u\,du + \int \sin u\,du$$

(Integrals No. 21, No. 20, No. 4 and No. 22)

$$= (-u^2 - 2u + 1)\cos u + 2(u + 1)\sin u$$

$$= (1 - x^2)\cos(x - 1) + 2x \sin (x - 1)$$

(b) $\int_0^2 x\,e^{2x}\,dx;$ $\qquad u = 2x$

$$\frac{1}{2}\,du = dx$$

$$\frac{1}{2}u = x$$

$\frac{1}{4}\int_0^4 u\,e^u\,du$ (Integral No. 16)

$$= \frac{1}{4}(ue^u - e^u)\Big|_0^4$$

$$= \frac{1}{4}(3e^4 + 1)$$

(c) $\int_0^{\pi} x \sin 3x \, dx;$ $\qquad u = 3x$

$$\frac{1}{3} du = dx$$

$$\frac{1}{3} u = x$$

$\frac{1}{9} \int_0^{3\pi} u \sin u \, du$ (Integral No. 20)

$$= \frac{1}{9}(-u \cos u + \sin u) \Big|_0^{3\pi}$$

$$= \frac{\pi}{3}$$

(d) $\int x \cos^3 (x^2) dx;$ $\qquad u = x^2$

$$du = 2x \, dx$$

$$\frac{1}{2} du = x \, dx$$

$\frac{1}{2} \int \cos^3 u \, du$ (Integral No. 29)

$$= \frac{\cos^2 u \sin u}{3} + \frac{2}{3} \sin u$$

$$= \frac{\cos^2 x^2 \sin x^2}{3} + \frac{2}{3} \sin x^2$$

(e) $\int x^3 e^{-4x};$ $\qquad u = -4x$

$$x^3 = -\frac{u^3}{64}$$

$$-\frac{1}{4} du = dx$$

$\frac{1}{256} \int u^3 e^u \, du$ (Integral No. 17)

$$= \frac{e^u}{256}[u^3 - 3u^2 + 6u - 6]$$

$$= \frac{-e^{-4x}}{128}[32x^3 + 24x^2 + 12x + 3]$$

(f) $\int xe^{x^2} \sin 2x^2 \, dx;$ $\qquad u = 2x^2$

$$x^2 = \frac{u}{2}$$

$$du = 4x \, dx$$

$\frac{1}{4} \int e^{u/2} \sin u \, du$ (Integral No. 24)

$$= \frac{1}{5} e^{u/2} (\frac{1}{2} \sin u - \cos u)$$

$$= \frac{1}{10} e^{x^2} (\sin 2x^2 - 2 \cos 2x^2)$$

(g) $\int_0^1 x^2 \log_e (x + 1) dx;$ $\qquad u = x + 1$

$$x^2 = (u - 1)^2$$

$$du = dx$$

$\int_1^2 (u^2 - 2u + 1) \log_e u \, du$ (Integral No. 19, No. 18 and No. 7)

$$= (\frac{u^3}{3} - u^2 + u) \log_e u + (- \frac{u^3}{9} + \frac{u^2}{2} - u) \Big|_1^2$$

$$= \frac{2}{3} \log_e 2 - \frac{5}{18}$$

(h) $\int \sin x \log_e (\cos x) dx;$ $\qquad u = \cos x$

$$du = -\sin x \, dx$$

$- \int \log u \, du$ (integral No. 7)

$$= -u \log_e u + u$$

$$= -\cos x \log_e \cos x + \cos x$$

(i) $\int \sin(x + 1)\cos(2x + 2)dx;$ $\quad u = x + 1$

$du = dx$

$\int \sin u \cos 2u \, du$ (Integral No. 32)

$$= \frac{\cos(-u)}{2} + \frac{\cos 3u}{6}$$

$$= \frac{\cos(-(x + 1))}{2} + \frac{\cos(3(x + 1))}{6}$$

$$\text{or} \quad \frac{\cos(x + 1)}{2} + \frac{\cos(3(x + 1))}{6}$$

(j) $\int \frac{\sin x}{2 \cos^2 x + \cos x - 3} dx;$ $\quad u = \cos x$

$du = -\sin x \, dx$

$-\int \frac{1}{2u^2 + u - 3} du;$ $\quad b^2 - 4ac > 0$

(Integral No. 36)

$$= \frac{1}{5} \log_e \frac{2u - 2}{2u + 3}$$

$$= \frac{1}{5} \log_e \frac{2 \cos x - 2}{2 \cos + 3}$$

(k) $\int \frac{e^x}{4e^{2x} - 2e^x + 1} dx;$ $\quad u = e^x$

$du = e^x \, dx$

$\int \frac{1}{4u^2 - 2u + 1} du;$ $\quad b^2 - 4ac < 0$

(Integral No. 37)

$$= \frac{1}{\sqrt{3}} \arctan \frac{4u - 1}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}} \arctan \frac{4e^x - 1}{\sqrt{3}}$$

(ℓ) $\dfrac{1}{x\sqrt{(\log_e x)^2 + 1}}\,dx;$ $\qquad u = \log_e x$

$$du = \frac{1}{x}\,dx$$

$\dfrac{1}{\sqrt{u^2 + 1}}\,du;$ (Integral No. 38)

$$= \log_e (u + \sqrt{u^2 + 1})$$

$$= \log_e(\log_e x + \sqrt{(\log_e x)^2 + 1})$$

7. $\displaystyle\int_0^1 e^{-x^2}\,dx \approx \alpha$

(a) The function e^{-x^2} is an even function since $f(x) = f(-x)$.

Thus $\displaystyle\int_{-1}^0 e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx = \alpha$

(b) $\displaystyle\int_{-1}^1 e^{-x^2}\,dx = \int_{-1}^0 e^{-x^2}\,dx + \int_0^1 e^{-x^2}\,dx$

$$= \alpha + \alpha = 2\alpha$$

(c) $\displaystyle\int_{-1}^3 e^{-\frac{(x-1)^2}{4}}\,dx;$ $\qquad u = \dfrac{x - 1}{2}$

$$du = \frac{1}{2}\,dx$$

$$2\,du = dx$$

$$2\int_{-1}^1 e^{-u^2}\,du = 2(2\alpha) = 4\alpha$$

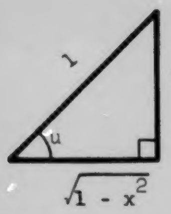
(d) $\displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx;$ $\qquad u = \sqrt{x}$

$$du = \frac{1}{2\sqrt{x}}\,dx$$

$$2\,du = \frac{1}{\sqrt{x}}\,dx$$

$$2\int_0^1 e^{-u^2}\,du = 2\alpha$$

8. (a) $\int_0^1 \sqrt{1 - x^2}\, dx; \quad u = \arcsin x$



Then $x = \sin u$

$dx = \cos u\, du$

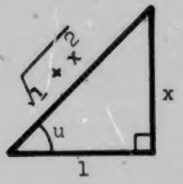
and $\sqrt{1 - x^2} = \cos u$

when $x = 0 \quad u = 0$

when $x = 1 \quad u = \frac{\pi}{2}$

$$\int_0^{\pi/2} \cos u \cos u\, du = \frac{u}{2} + \sin 2u \Big|_0^{\pi/2} = \frac{\pi}{4}$$

(b) $\frac{\sqrt{1 + x^2}}{x^4}\, dx; \quad u = \arctan x$



Then $x = \tan u$

$dx = \sec^2 u\, du$

and $\sqrt{1 + x^2} = \sec u$

$$\frac{\sec u \cdot \sec^2 u}{\tan^4 u}\, du = \frac{\cos u}{\sin^4 u}\, du$$

$v = \sin u$

$dv = \cos u\, du$

$$\int v^{-4}\, dv = \frac{v^{-3}}{-3}$$

$$= -\frac{1}{3} \csc^3 u$$

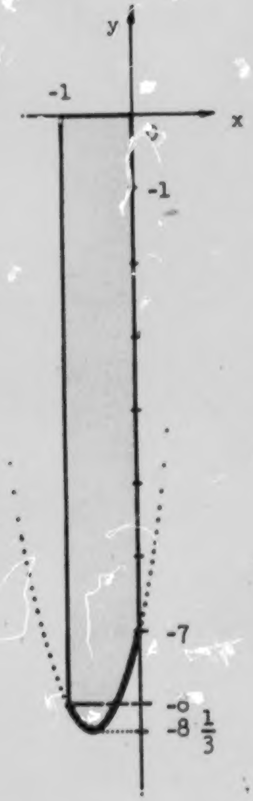
$$= -\frac{1}{3} \csc^3 \arctan x$$

$$= -\frac{1}{3} \frac{(1 + x^2)^{3/2}}{x^3}$$

Solutions Exercises 9-2

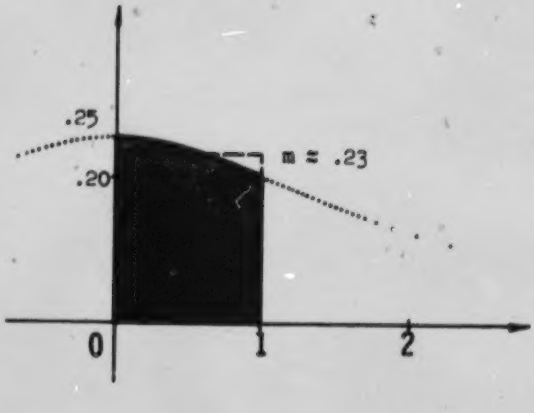
1. (a) $f : x \to 3x^2 + 4x - 7, \quad -1 \le x \le 0$

$$m = \frac{1}{0 - (-1)} \int_{-1}^{0} (3x^2 + 4x - 7) = -8$$



(b) $f : x \to \dfrac{1}{4 + x^2}, \quad 0 \le x \le 1$

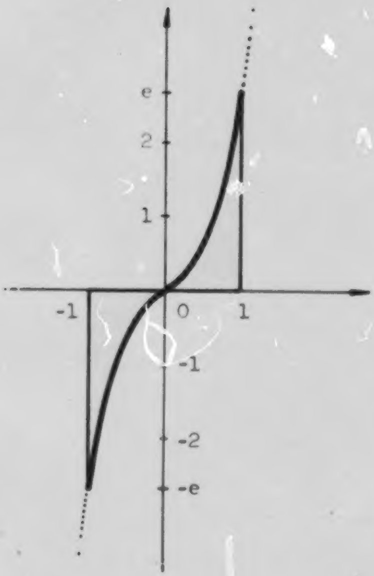
$$m = \frac{1}{1 - 0} \int_{0}^{1} \frac{1}{4 + x^2}\, dx \approx .23$$



(c) $f : s \to se^{s^2}$, $-1 \le s \le 1$

$$m = \frac{1}{1 - (-1)} \int_{-1}^{1} s\, e^{s^2}\, ds$$

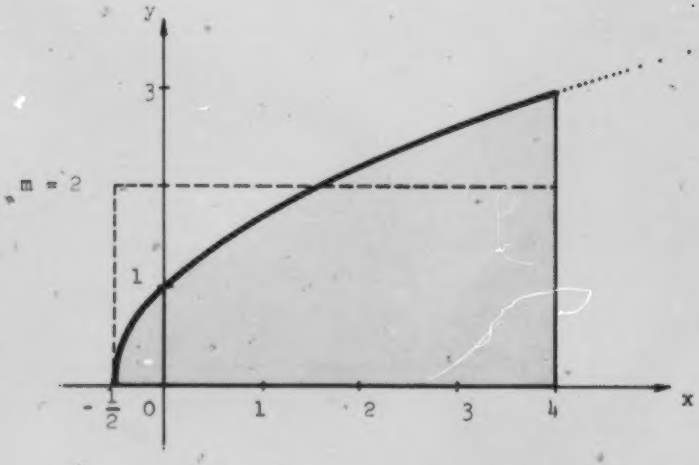
$$= \frac{1}{2}(0)$$



(d) $f : t \to \sqrt{2t + 1}$, $-\frac{1}{2} \le t \le 4$

$$m = \frac{1}{4 - (-\frac{1}{2})} \int_{-1/2}^{4} \sqrt{2t - 1}\, dt$$

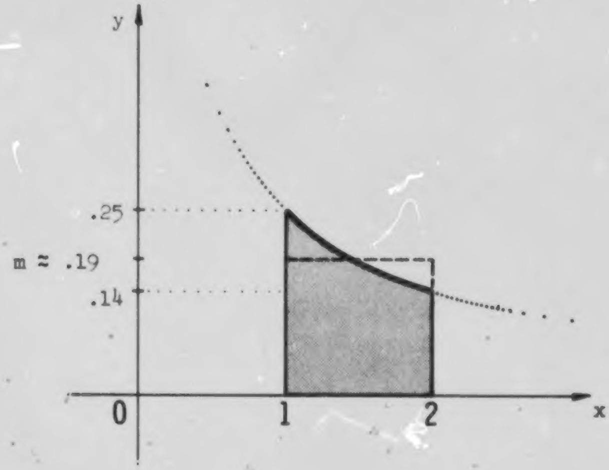
$$= \frac{2}{9}(9) = 2$$



(e) $f : x \to \frac{1}{3x+1}$, $1 \leq x \leq 2$

$$m = \frac{1}{2-1}\int_1^2 \frac{1}{3x+1}\,dx$$

$$\approx .19$$



2. $f : x \to \sin x$

(a) $0 \leq x \leq \pi$

$$m = \frac{1}{\pi}\int_0^\pi \sin x\,dx$$

$$= \frac{2}{\pi}$$

(b) $1 + 7\pi \leq x \leq 1 + 9\pi$

$$m = \frac{1}{(9\pi+1)-(7\pi+1)}\int_{1+7\pi}^{1+9\pi} \sin x\,dx$$

$$= 0$$

(c) $-\pi \leq x \leq \pi$

$$m = \frac{1}{\pi-(-\pi)}\int_{-\pi}^{\pi} \sin x\,dx$$

$$= 0$$

(d) $c \leq x \leq c + 2\pi$

$$m = \frac{1}{(c + 2\pi) - c} \int_{c}^{c+2\pi} \sin x \, dx$$

$$= 0$$

3. If f is periodic with period α then $f(x) = f(x + n\alpha)$, $n = 0, \pm 1, \pm 2, \ldots,$ for all x.

Let m be the average value for the interval $[x, x + \alpha]$

$$m = \frac{1}{(x + \alpha) - x} \int_{x}^{x+\alpha} f = \frac{1}{\alpha} [F(x + \alpha) - F(x)] \Big|_{x}^{x+\alpha}$$

where $F'(x) = f(x)$ and $F'(x + \alpha) = f(x + \alpha)$. This is true for all values of x, even for a special value of x such as $x = 0$. Thus

$$m = \frac{1}{\alpha} [F(\alpha) - F(0)]$$

which is a constant.

4. The average value of the slope of the tangent is simply $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.

In this case where $f : x \to x^2 + 1$ then for the interval $-1 \leq x \leq 3$,

$$m = \frac{10 - 2}{3 - (-1)} = 2.$$

Alternate solution:

The slope of $f : x \to x^2 + 1$ is $f'(x) = 2x$. The average value of the slope is

$$m = \frac{1}{3 - (-1)} \int_{-1}^{3} 2x \, dx$$

$$= \frac{1}{4} (8)$$

$$= 2$$

5. $f : x \to x^2$ on $0 \leq x \leq 1$

We examine the average value of f and the square root of the average value of f^2.

(i) The average value of f is

$$m_1 = \frac{1}{1} \int_0^1 x^2 \, dx$$

$$= \frac{1}{3}$$

(ii) The average value of f^2 is

$$m_2 = \frac{1}{1} \int_0^1 x^4 \, dx$$

$$= \frac{1}{5}$$

The average value of f is not the same as the square root of the average value of f^2. That is $m_1 \neq \sqrt{m_2}$.

6. The average acceleration is

$$m = \frac{1}{4 - 1} \int_1^4 (t^3 + \frac{1}{\sqrt{t}}) dt$$

$$= \frac{1}{3} \cdot \frac{263}{4} = \frac{263}{12} .$$

7. Let $f : x \to ax + b$ be a linear function. The average value of f on the interval $p \leq x \leq q$ is

$$m = \frac{1}{q - p} \int_p^q (ax + b) dx$$

$$= \frac{1}{q - p} [\frac{ax^2}{2} + bx] \Big|_p^q$$

$$= \frac{1}{q - p} [\frac{a}{2} q^2 + bq - \frac{a}{2} q^2 - bq]$$

$$= \frac{1}{(q - p)} [\frac{a}{2} (q^2 - p^2) + b(q - p)]$$

$$= \frac{a}{2}(q + p) + b$$

$$= (\frac{a}{2} p + \frac{b}{2}) + (\frac{a}{2} q + \frac{b}{2})$$

$$= \frac{f(p) + f(q)}{2}$$

8. If f is continuous on the interval $a \leq x \leq b$, by Theorem 8-2g f has at least one **maximum** on the interval, let us call it $f(x_1)$ and at least one minimum on the interval, let us call it $f(x_2)$. By Theorem 8-2g we also know that $a \leq x_1 \leq b$ and $a \leq x_2 \leq b$. Furthermore, the average value A is between $f(x_1)$ and $f(x_2)$. By the Intermediate Value Theorem, since f is continuous, there exists c, with c between x_2 and x_1 such that $f(c) = A$, the average value of f on $[a,b]$.

9. If $A = \frac{1}{c - a}\int_a^c f$ or $\int_a^c f = A(c - a)$,

$$B = \frac{1}{b - c}\int_c^b f \quad \text{or} \quad \int_c^b f = B(b - c)$$

and $C = \frac{1}{b - a}\int_a^b f$ then $c = \frac{1}{b - a}\int_a^c f + \frac{1}{b - a}\int_c^a f.$

By substitution

$$c = \frac{1}{b - a}(A(c - a)) + \frac{1}{b - a}(B(b - c))$$

and

$$c = \frac{c - a}{b - a} A + \frac{b - c}{b - a} B.$$

10. f is continuous on $a \leq x \leq b$ and $a \leq c \leq b$. By problem 7 the average f on $c - h \leq x \leq c + h$ is some $f(c_1)$ such that $c - h \leq c_1 \leq a + h$. But $\lim_{h \to 0} c - h = c$ and $\lim_{h \to 0} c + h = c$. Thus $c \leq c_1 \leq c$ and

$$f(c_1) = f(c) = \lim_{h \to 0} [\text{average of } f \text{ on } c - h \leq h \leq c + h].$$

11. (a) $$\lim_{n \to \infty} \frac{1}{n}[\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \dots \sin\frac{n\pi}{n}] = \frac{1}{1 - 0}\int_0^1 \sin \pi x$$
$$= \frac{2}{\pi}$$

(b) $$\lim_{n \to \infty} \frac{1^2 + 2^2 + 3^2 + \dots + n^2}{n^3} = \lim_{n \to \infty} \frac{1}{n}((\frac{1}{n})^2 + (\frac{2}{n})^2 + (\frac{3}{n})^2 + \dots + (\frac{n}{n})^2)$$
$$= \frac{1}{1 - 0}\int_0^1 x^2\,dx$$
$$= \frac{x^3}{3}\Big|_0^1$$
$$= \frac{1}{3}$$

(c) $$\lim_{n \to \infty} \frac{1^\alpha + 2^\alpha + 3^\alpha + \ldots + n^\alpha}{n^{\alpha+1}} = \lim_{n \to \infty} \frac{1}{n}\left(\left(\frac{1}{n}\right)^\alpha + \left(\frac{2}{n}\right)^\alpha + \ldots \left(\frac{n}{n}\right)^\alpha\right)$$

$$= \frac{1}{1 - 0} \int_0^1 x^\alpha \, dx$$

$$= \frac{x^{\alpha+1}}{\alpha + 1} \Bigg|_0^1$$

$$= \frac{1}{\alpha + 1}$$

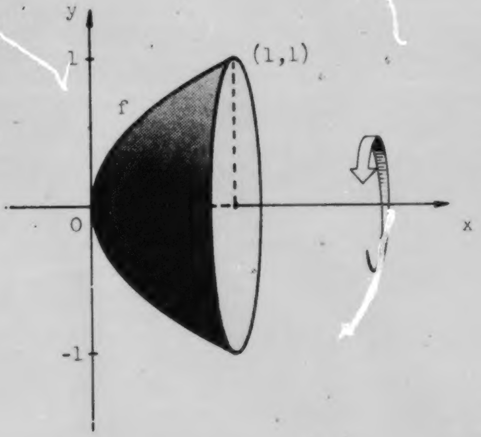
Solutions Exercises 9-3

1. Use (6) where $C = \pi x$.

$$V = (1 - 0) \times \left(\frac{1}{1 - 0}\right) \int_0^1 \pi \, x dx$$

$$= 1 \cdot \frac{\pi}{1} \cdot \frac{x^2}{2} \Big|_0^1$$

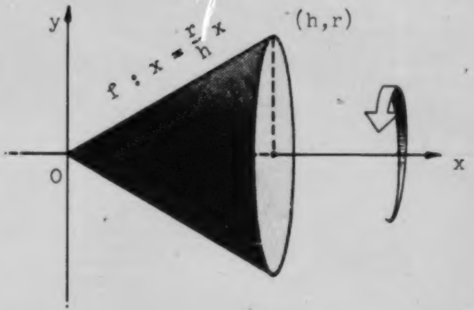
$$= \frac{\pi}{2}$$



2. $V = (h - 0) \dfrac{1}{h - 0} \displaystyle\int_0^h \pi \cdot \left(\frac{r}{h} x\right)^2 dx$

$$= h \cdot \frac{1}{h} \pi \frac{r^2}{h^2} \frac{x^3}{3} \Big|_0^h$$

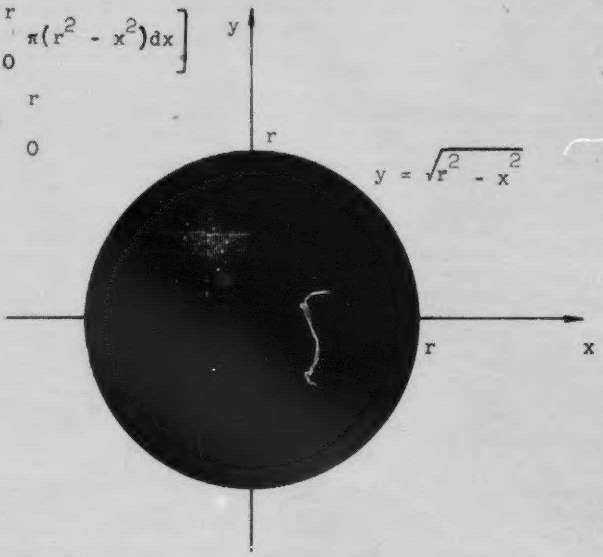
$$= \frac{\pi r^2}{h^2} \cdot \frac{h^3}{3} = \frac{1}{3} \pi r^2 h$$



3. $V = 2 \left[(r - 0) \cdot \dfrac{1}{(r - 0)} \displaystyle\int_0^r \pi(r^2 - x^2) dx \right]$

$$= 2 \cdot r \cdot \frac{1}{r} \cdot \pi\left(r^2 x - \frac{x^3}{3}\right) \Big._0^r$$

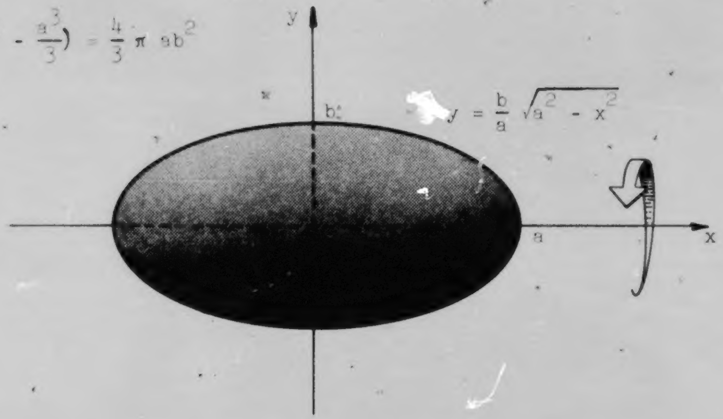
$$= 2\pi\left(r^3 - \frac{r^3}{3}\right) = \frac{4}{3} \pi r^3$$



4. $$V = 2\left[(a - 0) \cdot \frac{1}{(a - 0)} \int_0^a \pi \frac{b^2}{a^2} (a^2 - x^2)dx\right]$$

$$= 2 \cdot a \cdot \frac{1}{a} \cdot \pi \cdot \frac{b^2}{a^2} \left(a^2x - \frac{x^3}{3}\right) \Big|_0^a$$

$$= \frac{2\pi b^2}{a^2} \cdot \left(a^3 - \frac{a^3}{3}\right) = \frac{4}{3} \pi ab^2$$

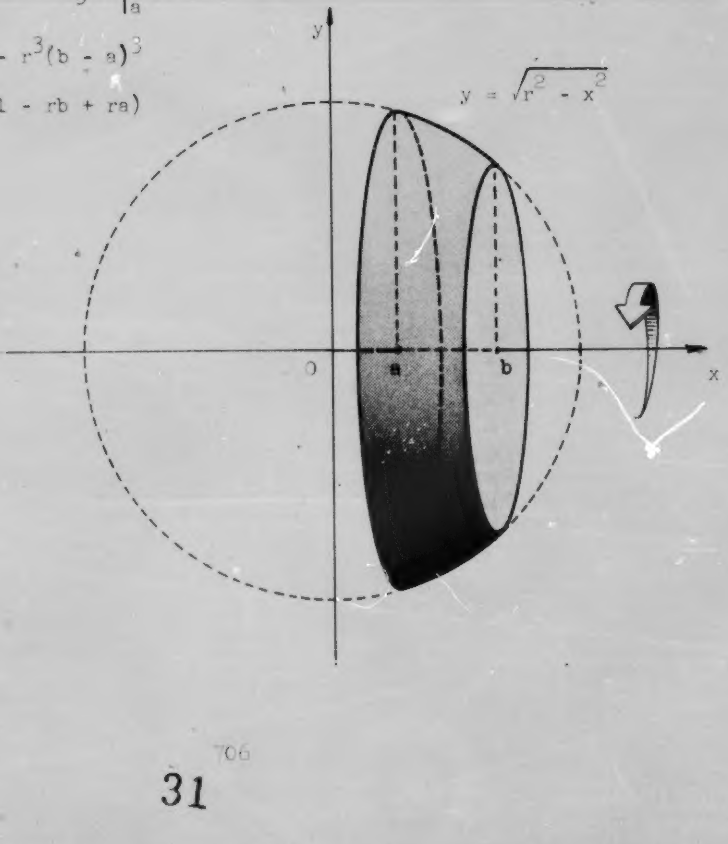

y
b
$y = \frac{b}{a}\sqrt{a^2 - x^2}$
a
x

5. $$V = b - a \cdot \frac{1}{b - a} \int_a^b \pi(r^2 - x^2)dx$$

$$= \frac{(b - a)}{b - a} \pi\left(r^2x - \frac{r^3}{3}\right) \Big|_a^b$$

$$= r^2(b - a) - r^3(b - a)^3$$

$$= r^2(b - a)(1 - rb + ra)$$


y
$y = \sqrt{r^2 - x^2}$
0
a
b
x

6. $V = V_1 - V_2$

where $f_1 : x = 2\sqrt{x}$

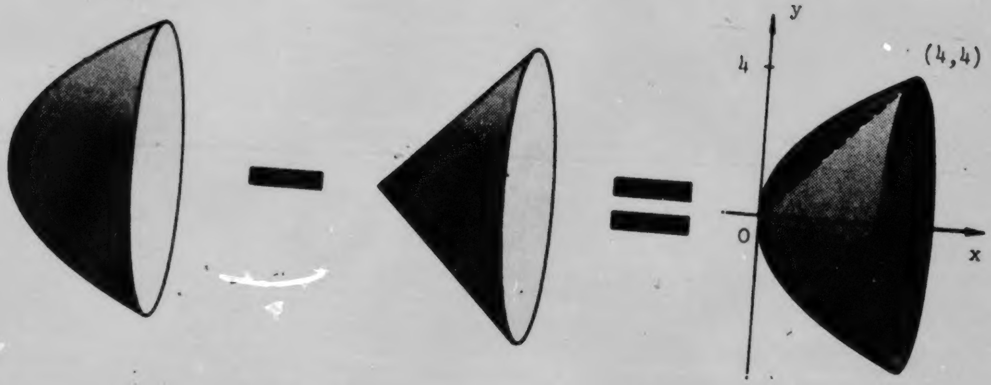
$f_2 : x = x$

$$V = 4 - 0 \cdot \frac{1}{4 - 0}\int_0^4 \pi(4x)dx - 4 - 0\,\frac{1}{4 - 0}\int_0^4 \pi(x^2)dx$$

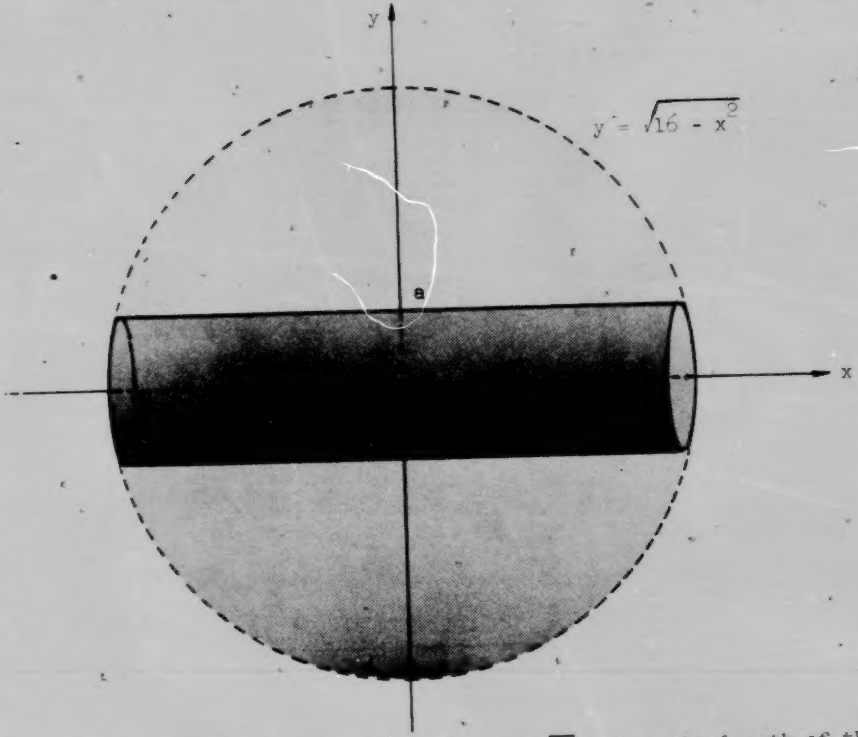
$$= \frac{4}{4} \cdot \pi \int_0^4 (4x - x^2)dx$$

$$= \pi(2x^2 - \frac{x^3}{3}) \Big|_0^4$$

$$= \pi(32 - \frac{6x}{3}) = \frac{32}{3}\pi$$



7. The required volume consists of a cylinder of 1 inch in radius and two "caps" each formed like a shallow dome.



The value of x for which $y = 1$ is $x = \sqrt{15}$. Thus the length of the cylinder is $2\sqrt{15}$ and its volume is $V_{cy} = \pi \cdot 1^2 \cdot 2\sqrt{15} = 2\pi\sqrt{15}$.

The volume of the 2 "caps" is found by

$$2V_{cap} = 2\left[(4 - \sqrt{15})\,\frac{1}{(4 - \sqrt{15})}\int_{\sqrt{15}}^{4} \pi(16 - x^2)dx\right]$$

$$= 2\pi(16x - \frac{x^3}{3})\Big|_{\sqrt{15}}^{4}$$

$$= 2\pi[(64 - \frac{64}{3}) - (16\sqrt{15} - 5\sqrt{15})]$$

$$= \pi[\frac{256}{3} - 22\sqrt{15}]$$

$$2V_{cap} + V_{cy} = \pi[\frac{256}{3} - 22\sqrt{15} + 2\sqrt{15}] = \pi[85\frac{1}{3} - 20\sqrt{15}]$$

$$\approx \pi(7.87366) \approx 24.7$$

It is interesting to note that the volume of the cylinder is $\approx 24.3^{+}$. Very little is contributed by the caps.

8. The cross-sectional area will be the difference of the areas of two circles. The larger will have a radius $y = \sqrt{r^2 - x^2}$, the smaller will have a constant radius of $\sqrt{r^2 - h^2}$. The cross-sectional area is given by

$$C(x) = \pi((r^2 - x^2) - (r^2 - h^2))$$

$$= \pi(h^2 - x^2)$$

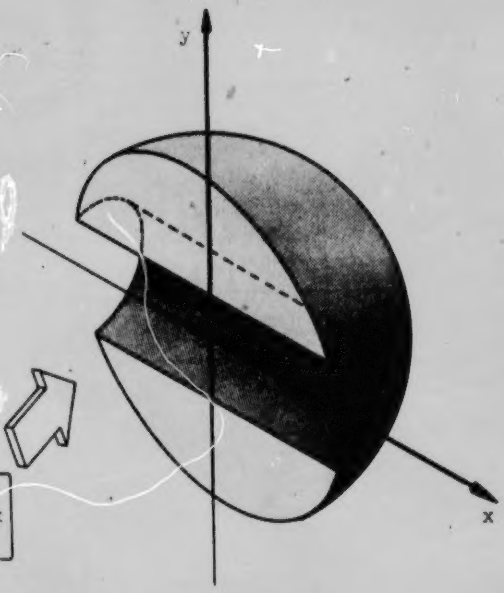
$$V = 2\left[(h - 0)\frac{1}{h - 0}\int_0^h \pi(h^2 - x^2)dx\right]$$

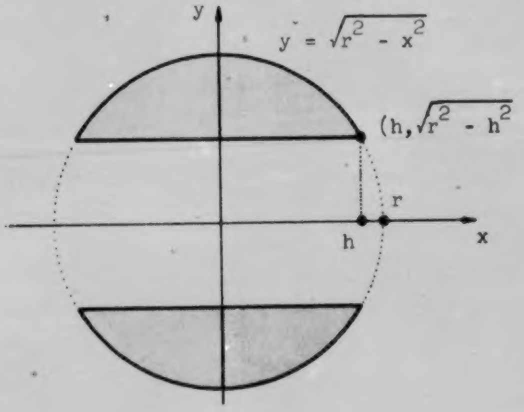
$$= 2\pi(h^2x - \frac{x^3}{3})\Big|_0^h$$

$$= 2\pi\frac{2h^3}{3} = \frac{4}{3}\pi h^3$$

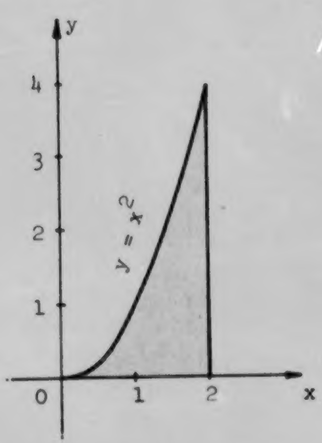
Special values such as $h = 0$ and $h = r$ yield

$$V_{h=0} = 0 \text{ and } V_{h=r} = \frac{4\pi}{3}r^3.$$





9. The volumes desired in this problem can be accomplished by translations of the shaded area in the figure. The lines about which we wish to rotate will be the axes in the rotated system.



(a) Let us define the translation

as $T : (x,y) \rightarrow (x, y + 4)$

Then $T : (y = x^2) \rightarrow y + 4 = x^2$

or $y = x^2 - 4.$

The cross-section is

$$C(x) = \pi[4^2 - (x^2 - 4)^2]$$

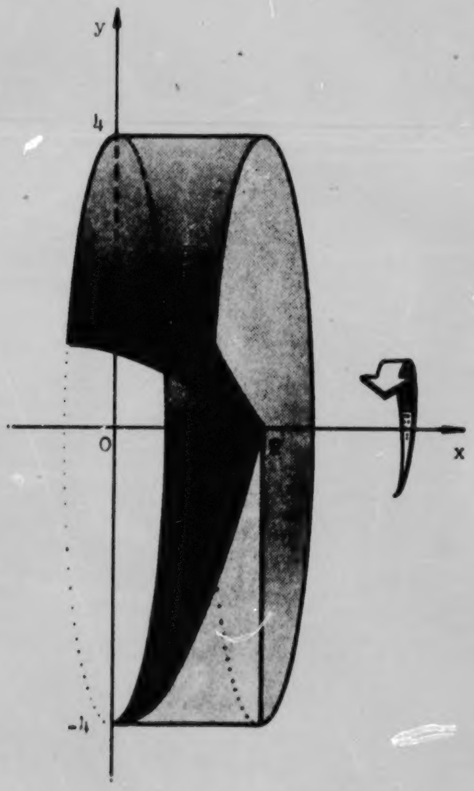
$$= \pi(-x^4 + 8x^2).$$

Thus V = length × average $C(x)$

$$= (2 - 0) \cdot \frac{1}{2 - 0} \int_0^2 \pi(-x^4 + 8x^2)dx$$

$$= 2\,\frac{1}{2} \cdot \pi\left(-\frac{x^5}{5} + \frac{8x^3}{3}\right)\Bigg|_0^2$$

$$= \pi\left(-\frac{32}{5} + \frac{64}{3}\right) = \frac{224}{15}\pi$$



(b) Let the transformation be $T : (x,y) \rightarrow (x,y - 2)$

Thus $T : (y = x^2) \rightarrow (y - 2 = x^2)$ or $y = x^2 + 2$.

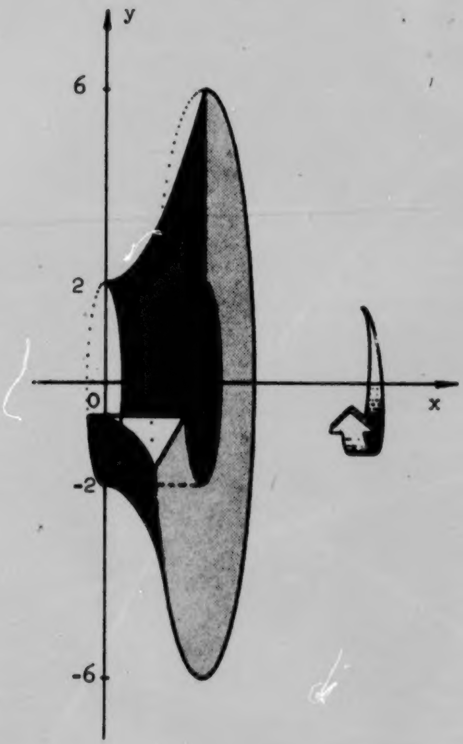
The average cross-section is

$$C(x) = \pi((x^2 + 2)^2 - 2^2)$$
$$= \pi(x^4 + 4x^2)$$

$$V = (2 - 0)\,\frac{1}{2 - 0}\int_0^2 \pi(x^4 + 4x^2)dx$$

$$= 2 \cdot \frac{\pi}{2}\left(\frac{x^5}{5} + \frac{4x^3}{3}\right)\Big|_0^2$$

$$= \pi\left(\frac{32}{5} + \frac{32}{3}\right) = \frac{256}{15}\pi$$



(c) Let $T : (x,y) \rightarrow (x + 2,y)$

$$y = (x + 2)^2$$

or $$x = \sqrt{y} - 2$$

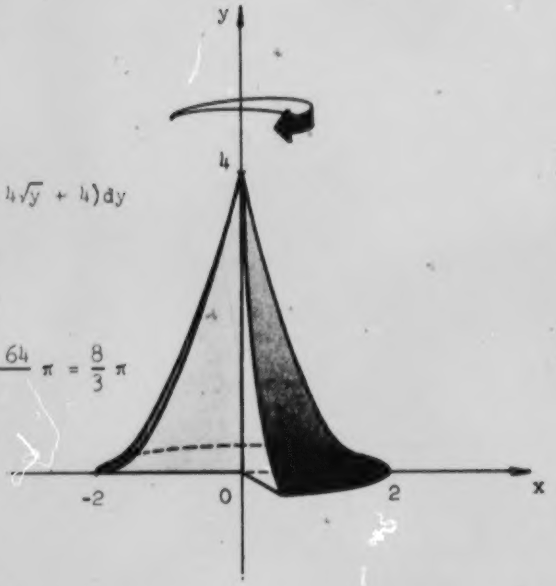
$$C(x) = \pi(\sqrt{y} - 2)^2$$

$$= \pi(y - 4\sqrt{y} + 4)$$

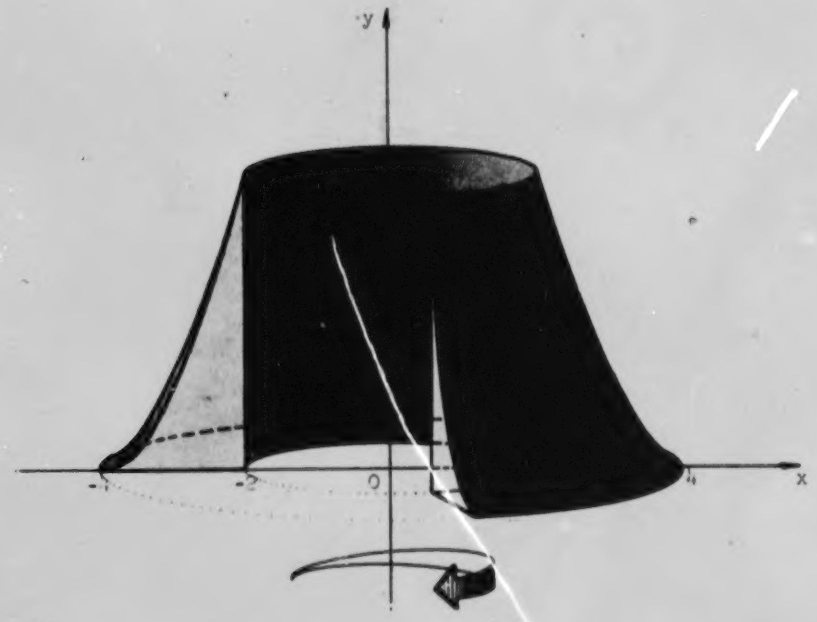
$$V = 4 - 0 \cdot \frac{1}{4 - 0}\int_0^4 \pi(y - 4\sqrt{y} + 4)dy$$

$$= \pi\left(\frac{y^2}{2} - \frac{4y^{3/2}}{\frac{3}{2}} + 4y\right)\Bigg|_0^4$$

$$= \pi\left(8 - \frac{64}{3} - 16\right) = \frac{72 - 64}{3}\pi = \frac{8}{3}\pi$$



Let the translation to $T : (x,y) \rightarrow (x + 4,y)$. Then $T : (y = x^2) \rightarrow (y = (x + 4)^2)$ and $x = \sqrt{y} - 4$



The cross-sectional area is

$$C(x) = \pi((\sqrt{y} - 4)^2 - 2^2)$$

$$= \pi(y - 8\sqrt{y} + 12)$$

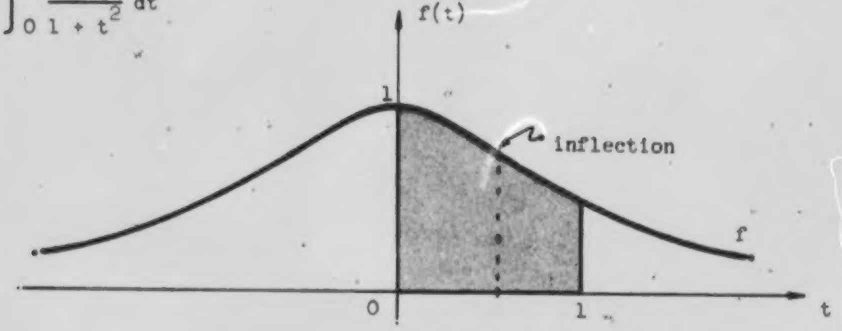
$$V = (4 - 0)\,\frac{1}{4 - 0}\int_0^4 \pi(y - 8\sqrt{y} + 12)dy$$

$$= 4 \cdot \frac{\pi}{4}\left(\frac{y^2}{2} - \frac{16y^{3/2}}{3} + 12y\right)\Big|_0^4$$

$$= \pi\left(8 - \frac{128}{3} + 48\right) = \frac{40}{3}\pi$$

Solutions Exercises 9-4

1. $\int_0^1 \frac{1}{1 + t^2}\,dt$



(a) If $n = 2$ $t_1 = 0,\ t_1 = \frac{1}{2},\ t_2 = 1$

t	f(t)
0	1
$\frac{1}{2}$	$\frac{4}{5}$
1	$\frac{1}{2}$

$$\int_0^1 f \approx \frac{1 - 0}{2 \cdot 2}\left(f(0) + 2f(\tfrac{1}{2}) + f(1)\right)$$

$$\approx \frac{1}{4}\left(1 + 2(\tfrac{4}{5}) + \tfrac{1}{2}\right) \approx \frac{31}{40} \approx 0.775$$

$$f'(t) = \frac{-2t}{(1 + t^2)^2}$$

$$f''(t) = \frac{2(3t^2 - 1)}{(1 + t^2)^3}$$

On the interval $[0,1]$, $f''(t)$ is maximum when $t = 0$. Then $|f''(0)| = |-2|$. Let $M = 1$ and the error is at most $\frac{2(1 - 0)^3}{12(2)^2} = \frac{1}{24} \approx .041\bar{6}$.

(b) If $n = 4$ then $t_0 = 0$, $t_1 = \frac{1}{4}$, $t_2 = \frac{1}{2}$, $t_3 = \frac{3}{4}$ and $t_4 = 1$.

t	$f(t)$
0	1
$\frac{1}{4}$	$\frac{16}{17} \approx 0.94$
$\frac{1}{2}$	$\frac{4}{5} = 0.80$
$\frac{3}{4}$	$\frac{16}{25} = 0.64$
1	$\frac{1}{2}$

$$\int_0^1 f(t)dt \approx \frac{1 - 0}{2 \cdot 4}(1 + 1.88 + 1.60 + 1.28 + 0.5)$$

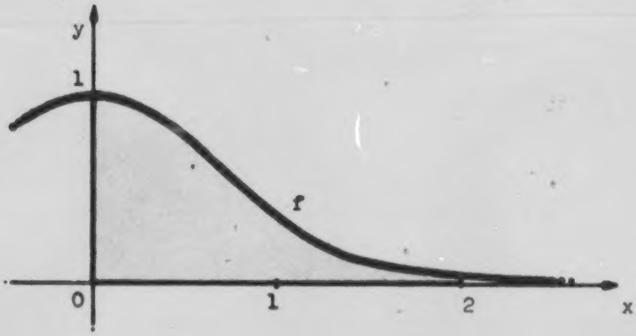
$$\approx \frac{3.13}{4} \approx 0.783$$

When $M = 1$ the maximum error is $\frac{2(1 - 0)^3}{12(4^2)} = \frac{1}{96} \approx 0.0104$.

The exact value of $\int_0^1 \frac{1}{1 + t^2} dt$ is found by integration.

$$\int_0^1 \frac{1}{1 + t^2} dt = \arctan t \Big|_0^1 = \frac{\pi}{4} \approx .785398 \approx .785$$

2. $\int_0^2 e^{-x^2} dx$



(a)

y	f
0	1.000
$\frac{1}{2}$	0.7788
1	0.3679
$\frac{3}{2}$	0.1054
2	0.0183

$$n = 2 \quad \int_0^2 e^{-x^2} dx \approx \frac{2 - 0}{2 \cdot 2}(1 + 0.7358 + 0.0183)$$

$$\approx \frac{1.7541}{2} \approx 0.8771$$

(b) $n = 4$ $$\int_0^2 e^{-x^2}\,dx \approx \frac{2-0}{2\cdot 4}(1 + 1.5576 + 0.7358 + 0.2108 + 0.0183)$$

$$\approx \frac{3.5225}{4} \approx 0.8806$$

To determine the maximum error first find f' and f''.

$$f' = -2xe^{-x^2} \quad \text{and} \quad f'' = 4x^2e^{-x^2}$$

$|f''|$ is maximum when $x = 2$.

$$|f''(2)| = |16e^{-4}| \approx 16(0.0183) \approx 0.2928$$

Let $M = 0.2928$, then the maximum error is $\dfrac{0.2928(2-0)^3}{12\cdot 2^2} \approx 0.0488$ when $n = 2$, or $\dfrac{0.2928(2-0)^3}{12\cdot 4^2} \approx 0.0122$ when $n = 4$.

3. $$\int_0^1 \frac{1}{1+t^2}\,dt$$

(a) $n = 1$ Use the values of Problem 1.

$$\int_0^1 \frac{1}{1+t^2}\,dt \approx \frac{1-0}{6\cdot 1}\left(1 + 4\left(\frac{4}{5}\right) + \frac{1}{2}\right)$$

$$\approx \frac{4.7}{6} \approx .78\overline{3}$$

This is the same as the actual value correct to two places.

(b) $n = 3$ Six such intervals are needed.

t	$f(t)$	multiple of $f(t)$	
$t_0 = 0$	1	1	1.000
$t_1 = \frac{1}{6}$	$\frac{36}{37}$	4	3.892
$t_2 = \frac{2}{6}$	$\frac{36}{40} = \frac{9}{10}$	2	1.800
$t_3 = \frac{3}{6}$	$\frac{36}{45} = \frac{4}{5}$	4	3.200
$t_4 = \frac{4}{6}$	$\frac{36}{52} = \frac{9}{13}$	2	1.385
$t_5 = \frac{5}{6}$	$\frac{36}{61}$	4	2.361
$t_6 = \frac{6}{6}$	$\frac{36}{72}$	1	.500

sum 14.138

$$\int_0^1 \frac{1}{1 + t^2}\,dt \approx \frac{1 - 0}{6 \cdot 3}(14.138)$$

$$\approx .785445 \approx .7854$$

This is the same as the actual value correct to four places. (Refer to Problem 1.)

The estimate of error requires finding the third and fourth derivatives.

$$f'(t) = \frac{-2t}{(1 + t^2)^2}$$

$$f''(t) = \frac{2(3t^2 - 1)}{(1 + t^2)^3}$$

$$f'''(t) = \frac{24t(1 - t^2)}{(1 + t^2)^4}$$

$$f^{(4)}(t) = \frac{12(t^4 - 12t^2 + 2)}{1 + t^2}$$

$$f^4(1) = 54 = M$$

Then the error $\leq \dfrac{54(1 - 0)^5}{180(2 \cdot 1)^4} \approx .03750$ for $n = 1$, and

the error $\leq \dfrac{54(1 - 0)^5}{180(2 \cdot 3)^4} \approx 0.0002315$ for $n = 3$.

4. $\displaystyle\int_0^2 e^{-x^2}\,dx$

x	e^{-x^2}	$n = 1$	$n = 2$
0	1	$f(1) = 1.0000$	$f(1) = 1.0000$
$\frac{1}{2}$	0.7788		$4f(\frac{1}{2}) = 3.1152$
1	0.3679	$4f(1) = 1.4716$	$2f(1) = .7358$
$\frac{3}{2}$	0.1054		$4f(\frac{3}{2}) = .4216$
2	0.0183	$f(2) = 0.0366$	$f(2) = 0.0366$
	Sum	2.5082	5.3092

(a) $\int_0^2 e^{-x^2}\,dx \approx \frac{2-0}{6 \cdot 1}(2.5082)$ when $n = 1$

$\approx .83606 \approx .84$

(b) $\int_0^2 e^{-x^2}\,dx \approx \frac{2-0}{6 \cdot 2}(5.3092)$ when $n = 2$

$\approx .884868 \approx .885$

To estimate the error we first find M.

$$f' = -2xe^{x^2}$$

$$f'' = 2e^{-x^2}(2x^2 + 1)$$

$$f''' = 16x^3 e^{-x^2}$$

$$f^{(4)} = 16x^2 e^{-x^2}(2x^2 - 3)$$

$f^{(4)}$ is its greatest at $x = \frac{1}{2}$ on the interval $0 \leq x \leq 2$. [Note: $x = \frac{1}{2}$ is a zero of $f^{(5)}$.]

$$f^{(4)}(0) = 0$$

$$f^{(4)}(\tfrac{1}{2}) \approx 7.788$$

$$f^{(4)}(1) \approx 5.886$$

$$f^{(4)}(\tfrac{3}{2}) \approx 5.692$$

$$f^{(4)}(2) \approx 1.171$$

Let $M = 8$ then the error is less than

$$\frac{8(2-0)^5}{180(2 \cdot 1)^4} \approx .0\overline{8} \quad \text{when } n = 1.$$

and less than

$$\frac{8(2-1)^5}{180(2 \cdot 2)^4} \approx .00\overline{5} \quad \text{when } n = 2.$$

5. Let $f : x \to Ax^3 + Bx^2 + Cx + D$ where $B = C = D = 0$.

The area $\int_a^b Ax^3 = \frac{A}{4}(b^4 - a^4)$ by actual integration.

By Simpsons Rule

$$\int_a^b Ax^3 = \frac{b - a}{6}\,[f(a) + 4f\frac{(a + b)}{2} + f(b)]$$

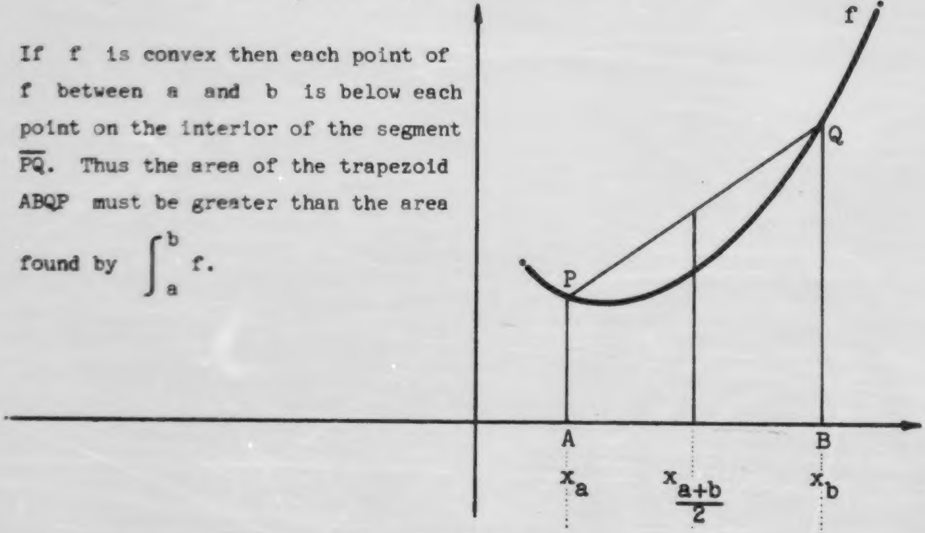
$$= \frac{A(b - a)}{6}\,[a^3 + 4\,(\frac{a + b}{2})^3 + b^3]$$

$$= \frac{A(b - a)}{6}\,[a^3 + \frac{1}{2}(b^3 + 3b^2a + 3ba^2 + a^3) + b^3]$$

$$= \frac{A}{4}\,(b - a)(b^3 + b^2a + ba^2 + b^3)$$

$$= \frac{A}{4}(b^4 - a^4)$$

6. If f is convex then each point of f between a and b is below each point on the interior of the segment $\overline{PQ}$. Thus the area of the trapezoid ABQP must be greater than the area found by $\int_a^b f$.



7. In the case of the Trapezoid Rule (3) the number n is the number of subintervals into which [a,b] is partitioned. Whereas in the case of Simpsons Rule (8) the interval [a,b] is actually partitioned into 2n subintervals.

In (3) n represents the number of trapezoidal regions and in (8) n represents the number of parabolic regions into which f is divided over the domain [a,b].

8. From problem 3 we found that $54 \leq M$. In order to insist on accuracy to five decimal places, n must satisfy this inequality.

$$\frac{54(1 - 0)^5}{180(2n)^4} \leq .000005$$

Solving for an integer value of n,

$$\frac{3}{160n^4} \leq \frac{1}{2 \times 10^5}$$

$$\frac{3 \times 10^4}{8} \leq n^4$$

$$61.24 \leq n^2$$

and $8 \leq n$ will do quite well.

9. $\log_e 3 = \int_1^3 \frac{1}{x}\,dx$

(i) Trapezoidal Rule:

If we are to insure four decimal place accuracy then

$$\frac{M(3 - 1)^3}{12n^2} \leq \frac{1}{2 \times 10^4}.$$

$$f : x \to \frac{1}{x}$$

$$f': x \to -\frac{1}{x^2}$$

$$f'': x \to 2\,\frac{1}{x^3}.$$

Let $2(\frac{1}{1}) \leq M$.

$$\frac{2(2)^3}{12n^2} \leq \frac{1}{2 \times 10^4}$$

$$\frac{2^5 \times 10^4}{12} \leq n^2$$

$$163 < n$$

We had better choose a different method!

(ii) Simpsons Rule:

$$f''' = -6\,\frac{1}{x^4}$$

$$f^{(4)} = 24\,\frac{1}{x^5}$$

$$M = 24$$

The required value of n must satisfy this inequality

$$\frac{24(3-1)^5}{180(2n)^4} \le \frac{1}{2 \times 10^4}$$

$$\frac{24 \cdot 2^5 \cdot 2 \cdot 10^4}{180 \cdot 2^4} \le n^4$$

$$\frac{24 \cdot 2 \cdot 10^3}{18\ 9\ 3}$$

$$\frac{16 \cdot 10^3}{3} \le n^4$$

$$5{,}333 \le n^4$$

$$73 \le n^2$$

$$9 \le n$$

We can let $n = 10$. This means that we must partition the interval $[1,3]$ into $2n = 20$ subintervals.

x	$\frac{1}{x}$	Multiple of $\frac{1}{x}$	
1.0	1.00000	1	1.00000
1.1	.90909	4	3.63636
1.2	.83333	2	1.66667
1.3	.76923	4	3.07692
1.4	.71428	2	1.42856
1.5	.66667	4	2.66667
1.6	.62500	2	1.25000
1.7	.58824	4	2.35296
1.8	.55556	2	1.11111
1.9	.52632	4	2.10528
2.0	.50000	2	1.00000
2.1	.47619	4	1.90476
2.2	.45455	2	.90909
2.3	.43478	4	1.73912
2.4	.41667	2	.83333
2.5	.40000	4	1.60000
2.6	.38462	2	.76923
2.7	.37037	4	1.48148
2.8	.35714	2	.71428
2.9	.34483	4	1.37932
3.0	.33333	1	.33333
			32.95847
			SUM

$$\int_1^3 \frac{1}{x}\,dx \approx \frac{(3-1)}{6\cdot 10}(32.95847)$$

$$\approx 1.09860$$

The actual value of $\log_e 3$ is 1.09861. Our answer turns out to be correct to 4 places as directed.

Solutions Exercises 9-5

1. $-1 \leq \cos x \leq 1$

(a) $\int_0^x -dt \leq \int_0^x \cos t\, dt \leq \int_0^x dt$

$-x \leq \sin x \leq x$

(b) $\int_0^x -t\, dt \leq \int_0^x \sin t\, dt \leq \int_0^x t\, dt$

$-\frac{x^2}{2} \leq 1 - \cos x \leq \frac{x^2}{2}$

(c) $\int_0^x -\frac{t^2}{2}\, dt \leq \int_0^x (1 - \cos t)dt \leq \int_0^x \frac{t^2}{2}\, dt$

$-\frac{x^3}{3!} \leq x - \sin x \leq \frac{x^3}{3!}$

(d) $\int_0^x -\frac{t^3}{3!}\, dt \leq \int_0^x (t - \sin t)dt \leq \int_0^x \frac{t^3}{3!}\, dt$

$-\frac{x^4}{4!} \leq \cos x - (1 - \frac{x^2}{2}) \leq \frac{x^4}{4!}$

(e) $\int_0^x -\frac{t^4}{4!} \leq \int_0^x (\cos t - 1 + \frac{t^2}{2})dt \leq \int_0^x \frac{t^4}{4!}\, dt$

$-\frac{x^5}{5!} \leq \sin x - (x - \frac{x^3}{3!}) \leq \frac{x^5}{5!}$

2. If $x \leq 0$ we will change each result depending on whether it is an even or odd function. If f is even then $f(x) = f(-x)$, whereas if f is odd then $f(x) = -f(-x)$.

(a) $-(-x) \leq \sin(-x) \leq (-x)$

$x \leq -\sin x \leq -x, \quad x \leq 0$

(b) $-\frac{x^2}{2} \leq 1 - \cos x \leq \frac{x^2}{2}$, for all x. These are even functions thus $f(x) = f(-x)$.

(c) $-\frac{(-x)^3}{3!} \leq (-x) - \sin(-x) \leq \frac{(-x)^3}{3!}$

$\frac{x^3}{3!} \leq -x + \sin x \leq -\frac{x^3}{3!}, \ x \leq 0.$

(d) $-\frac{x^4}{4!} \le \cos x - (1 - \frac{x^2}{2!}) \le \frac{x^4}{4!}$, for all x.

(e) $-\frac{(-x)^5}{5!} \le \sin(-x) - ((-x) - \frac{(-x)^3}{3!}) \le \frac{(-x)^5}{5!}$

$$\frac{x^5}{5!} \le -\sin x + (x + \frac{x^3}{3!}) \le -\frac{x^5}{5!}, \quad x \le 0$$

3. $f : x \to \sqrt[3]{1 + x}$ $\qquad f(0) = 1$

$f': x \to \frac{1}{3}(1 + x)^{-2/3}$ $\qquad f'(0) = \frac{1}{3}$

$f'': x \to -\frac{1}{3} \cdot \frac{2}{3}(1 + x)^{-5/3}$ $\qquad f''(0) = -\frac{2}{9}$

$f''': x \to \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{5}{3}(1 + x)^{-8/3}$ $\qquad f'''(0) = \frac{10}{27}$

$f^{(4)}: x \to -\frac{1}{3} \cdot \frac{2}{3} \cdot \frac{5}{3} \cdot \frac{8}{3}(1 + x)^{-11/3}$ $\qquad f^{(4)}(0) = -\frac{80}{81}$

$$p_3(x) = 1 + \frac{1}{3}x - \frac{2}{9} \cdot \frac{x^2}{2!} + \frac{10}{27} \cdot \frac{x^3}{3!}$$

$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3$$

$$f(x) = p_3(x) + R_3$$

On the interval $[0,1]$, $|f^{(4)}(x)|$ is a maximum when $x = 0$ and $|f^4(0)| = \frac{80}{81} \le K$. From (16), $|R_3| \le K\frac{|x|^4}{4!} \le K\frac{M^4}{4!}$, $0 \le x \le 1$. Let $x = 1$ since this is the maximum value of x on the interval $[0,1]$. Then

$$|R_3| \le \frac{80}{81} \cdot \frac{1}{4!} = \frac{10}{243} \approx .04115.$$

4. (a) $f : x \to \sqrt{1 + x}$, x on the interval $-1 < x \le 0$

$f^4: x \to -\frac{15}{16}(1 + x)^{-7/2}$

$|f^{(4)}(x)| \le K$

K is maximum when $x \to -1$.

$$\lim_{x \to -1} f^{(4)}(x) = \infty$$

Thus $|R_n| \le K\frac{|x|^{n+1}}{(n + 1)!}$

$$|R_3| \le \lim_{x \to -1} f^4(x)\frac{(1)^4}{4!} \to \infty$$

There is no error estimate possible near $x = -1$.

(b) A more realistic problem is to estimate error over a closed interval. We selected $-.5 \leq x \leq 0$.

$$f^{(4)}(-.5) = -\frac{15}{16} \cdot\cdot 2^{7/2} \qquad f^{4}(0) = -\frac{15}{16}$$

$$|f^{(4)}(x)| \leq \frac{15}{2}\sqrt{2}$$

Thus $|R_3| \leq \frac{15}{2}\sqrt{2}\,\frac{|-.5|^4}{4!} = \frac{5\sqrt{2}}{256} \approx .02762$.

5. (a) $f : x \to \frac{1}{1+x}$ $\qquad f(0) = 1$

$f' : x \to \frac{-1}{(1+x)^2}$ $\qquad f'(0) = -1$

$f'' : x \to \frac{2!}{(1+x)^3}$ $\qquad f''(0) = 2!$

$\vdots \qquad\qquad \vdots$

$f^{(n-1)} : x \to \frac{(n-1)!}{(1+x)^n}$ $\qquad f^{(n-1)}(0) = (-1)^{n-1}(n-1)!$

$$p_{n-1}(x) = 1 - x + \frac{2!}{2!}x^2 - \frac{3!}{3!}x^3 + \ldots (-1)^{(n-1)}\frac{(n-1)!}{(n-1)!}x^{n-1}$$

$$= 1 - x + x^2 - x^3 + \ldots (-1)^{n-1}x^{n-1} + R_{n-1}$$

(b) $|R_{n-1}| \frac{|x|^n}{|1+x|}$ then $|R_n| \leq \frac{|x|^{n+1}}{|x+1|}$

(c) For $p_5(10)$, $R_n \leq \frac{10^6}{11} \approx 909{,}091$

This is very large as should be expected.

(d) $\lim\limits_{n \to \infty} \frac{1}{1+x} \to 0$ whereas $\lim\limits_{n \to \infty} |R_n| \to \infty$.

When $x > 1$. Since $p_n(x)$ is an alternating series, $p_n(x)$ oscillates wildly as $n \to \infty$.

6. $f : x \to \frac{1}{2 + x} = \frac{1}{2}(\frac{1}{1 + \frac{x}{2}})$ $\qquad f(0) = \frac{1}{2}$

$f': x \to \frac{1}{2} \cdot \frac{1}{2} \frac{-1}{(1 + \frac{x}{2})^2}$ $\qquad f'(0) = (\frac{1}{2})^2$

$f'': x \to \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{2!}{(1 + \frac{x}{2})^3}$ $\qquad f''(0) = (\frac{1}{2})^3\ 2!$

$f''': x \to \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{-3!}{(1 + \frac{x}{2})^4}$ $\qquad f'''(0) = -(\frac{1}{2})^4\ 3!$

$\vdots \qquad\qquad \vdots$

$f^{(n-1)}: x \to (\frac{1}{2})^{n-1}(-1)^n \frac{n!}{(1 + \frac{x}{2})^{n+1}}$ $\qquad f^{(n-1)}(0) = (-1)^n(\frac{1}{2})^{n-1}n!$

$$P_{n-1}(x) = \frac{1}{2} - (\frac{1}{2})^2 x + (\frac{1}{2})^3 x^2 - (\frac{1}{2})^4 x^3 \dots (-1)^n(\frac{1}{2})^{n-1} x^{n-1}$$

$$= \frac{1}{2}(1 - \frac{x}{2} + (\frac{x}{2})^2 - (\frac{x}{2})^3 + \dots (-1)^n(\frac{x}{2})^{n-1})$$

$$R_{n-1} = \frac{(\frac{x}{2})^n}{2(1 + \frac{x}{2})} = \frac{(\frac{x}{2})^n}{(2 + x)}$$

As $n \to \infty$ the only values of x for which $R_{n-1} \to 0$ is when $|\frac{x}{2}| < 1$ or $|x| < 2$.

7. $f : x \to \log_e(2 + x)$ $\qquad f(0) = \log_e 2$

$f': x \to \frac{1}{2 + x} = \frac{1}{2(1 + \frac{x}{2})}$ $\qquad f'(0) = \frac{1}{2}$

$f'': x \to \frac{1}{2} \cdot \frac{1}{2} \frac{-1}{(1 + \frac{x}{2})^2}$ $\qquad f''(0) = -(\frac{1}{2})^2$

$f''': x \to \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \frac{2!}{(1 + \frac{x}{2})^2}$ $\qquad f'''(0) = (\frac{1}{2})^3 \cdot 2!$

$\vdots \qquad\qquad \vdots$

$f^{(n-1)}(x) \to (-\frac{1}{2})^{n-1} \frac{(n-1)!}{(1 + \frac{x}{2})^n}$ $\qquad f^{(n-1)}(0) = (-\frac{1}{2})^{n-1} \cdot (n-1)!$

$$P_{n-1}(x) = \log_e 2 + \frac{x}{2} - \frac{1}{2}(\frac{x}{2})^2 + \frac{1}{3}(\frac{x}{2})^3 + \dots \frac{(-1)^{n-2}}{n-1}(\frac{x}{2})^{n-1}$$

$$R_{n-1} = \frac{(-1)^{n-1}(\frac{x}{2})^n}{n(1 + \frac{x}{2})}$$

The limit of R_{n-1} as $n \to \infty$ is dependent upon $(\frac{x}{2})^n$. If $|\frac{x}{2}| > 1$ then $\lim\limits_{n \to \infty} R_{n-1} \to \infty$; if $|\frac{x}{2}| < 1$ then $\lim\limits_{n \to \infty} R_{n-1} \to 0$. Thus, $0 < x < 2$ is necessary for $R_{n-1} \to 0$ as $n \to \infty$.

8. (a) $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

Let $\alpha = \arctan\frac{1}{2}$

and $\beta = \arctan\frac{1}{3}$

Then $\tan(\alpha + \beta) = \dfrac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2}\cdot\frac{1}{3}} = \dfrac{\frac{5}{6}}{\frac{5}{6}} = 1.$

Thus $\alpha + \beta = \frac{\pi}{4}$.

(b) $\pi = 4(\arctan\frac{1}{2} + \arctan\frac{1}{3})$

The remainder term (25) gives us our estimate of error.

$$|R_n| = |(-1)^n \int_0^x \frac{t^{2n}}{1 + t^2}\,dt| \le |\int_0^x t^{2n}dt|$$

$$|R_n| \le \frac{x^{2n+1}}{2n + 1}$$

Since two decimal place accuracy is required then

$$4(R_n(\arctan\tfrac{1}{2}) + R_n(\arctan\tfrac{1}{3})) \le 5 \times 10^{-3}$$

also $$R_n(\arctan\tfrac{1}{2}) \le \frac{(\frac{1}{2})^{2n+1}}{2n + 1}$$

and $$R_n(\arctan\tfrac{1}{3}) \le \frac{(\frac{1}{3})^{2n+1}}{2n + 1}$$

finally $$\frac{4}{2n + 1}[(\tfrac{1}{2})^{2n+1} + (\tfrac{1}{3})^{2n+1}] \le 5 \times 10^{-3}.$$

We simplify this inequality to obtain

$$8 \times 10^2(3^{2n+1} + 2^{2n+1}) \le (2n + 1)6^{2n+1}.$$

Try $n = 3$; $1,852,000 \le 1,959,552$. Since $n = 3$ just barely satisfies the inequality we will use $n = 4$ to insure success.

$$\pi = 4[\arctan \tfrac{1}{2} + \arctan \tfrac{1}{3}]$$

$$\approx 4[\tfrac{1}{2} - \tfrac{1}{24} + \tfrac{1}{160} - \tfrac{1}{896} + \tfrac{1}{3} - \tfrac{1}{81} + \tfrac{1}{1215} - \tfrac{1}{15309}]$$

$\approx 3.14096 = 3.14$ to two decimal places.

9. (a) Let $\alpha = 4 \arctan \frac{1}{5}$ and $\beta = \arctan \frac{1}{239}$.

Let $\theta = \arctan \frac{1}{5}$

$$\tan \alpha = \tan(2\theta + 2\theta)$$

$$\tan 2\theta = \frac{2 \cdot \frac{1}{5}}{1 - \frac{1}{25}} = \frac{5}{12}$$

$$\tan \alpha = \tan 2(2\theta) = \frac{2 \cdot \frac{5}{12}}{1 - \frac{25}{144}} = \frac{120}{119}$$

$$\tan(\alpha - \beta) = \frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \cdot \frac{1}{239}} = \frac{\frac{28561}{28441}}{\frac{28561}{28441}} = 1$$

Thus $(\alpha - \beta) = \frac{\pi}{4}$.

(b) Before actually calculating π we must investigate the error estimate.

Let R be the error estimate for the Taylor approximation of $\arctan \frac{1}{5}$ and S be the error estimae for $\arctan \frac{1}{239}$. The total error estimate must be less than 5×10^{-3}. Since $4 \cdot (\frac{4}{\pi}) = 4[4 \arctan \frac{1}{5} - \arctan \frac{1}{239}]$ the maximum error will be $4(4|R| + |S|)$. We do not have to use the same n for both R and S.

$$|R| \le \left| \frac{(\frac{1}{5})^{2n+1}}{2n+1} \right| \quad \text{and} \quad |S| \le \left| \frac{(\frac{1}{239})^{2n+1}}{2n+1} \right|$$

As a trial let $n = 1$ to calculate S.

$$|S|_{n=1} \le \frac{(\frac{1}{239})^3}{3} \approx 2 \cdot 4 \times 10^{-8}$$

Let us try $n = 3$ for R.

$$|R| \leq \frac{(\frac{1}{5})^7}{7} \approx 1.828 \times 10^{-6}$$

$$4|R| \leq 7.312 \times 10^{-6} \approx 731.2 \times 10^{-8}$$

$$4|R| + |S| \leq 733.6 \times 10^{-8} \approx 8 \times 10^{-6}$$

The total error estimate can now be calculated.

$$4[4|R| + |S|] \leq 3.2 \times 10^{-5} < 5 \times 10^{-2}$$

$$\pi \approx 4[4(\frac{1}{5} - \frac{1}{3}(\frac{1}{5})^3 + \frac{1}{5}(\frac{1}{5})^5) - (\frac{1}{239})]$$

$$\approx 3.14152 \approx 3.14 \text{ to two decimal places.}$$

Obviously, we used more terms than were necessary.

10. (a) Show that $\log_e 2 = -7 \log_e \frac{9}{10} + 2 \log_e \frac{24}{20} + 3 \log_e \frac{81}{80}$.

This is true if $2 = (\frac{9}{10})^7 \cdot (\frac{24}{25})^2 \cdot (\frac{81}{80})^3$

$$= \frac{2^{13} \cdot 5^7 \cdot 3^{14}}{2^{12} \cdot 5^7 \cdot 3^{14}} = 2$$

(b) Each number $\frac{9}{10}$, $\frac{24}{25}$ and $\frac{81}{80}$ is very close to 1. The greatest error in any one of these Taylor approximations of the logarithms will occur for $x = -\frac{1}{10}$, call this error R.

The maximum possible accumulated error obtainable from summing the three series will be greater than $|R| + 2|R| + 3|R| = 12|R|$. If five decimal place accuracy is required then

$$12|R| \leq 5 \times 10^{-6}$$

must be satisfied.

Let $$|R| \leq \frac{(\frac{1}{10})^{n+1}}{n+1} \text{ from (23).}$$

Then $$\frac{12(\frac{1}{10})^{n+1}}{n+1} \leq 5 \times 10^{-6} < 6 \times 10^{-6}$$

or $$\frac{2}{10^{n-5}} < n + 1.$$

We find that $n \geq 5$ is the integer solution of this inequality.

11. Since $f : x \rightarrow \log \frac{1 + x}{1 - x}$ then also $f : x \rightarrow \log_e(1 + x) - \log_e(1 - x)$

$$p_n(x) \approx (x - \frac{x^2}{2} + \frac{x^3}{3} + \dots (-1)^{n-1} \frac{x^n}{n} + (R_1)_n) - (-x - \frac{x^2}{2} - \frac{x^3}{3} + \dots$$

$$- \frac{x^n}{n} + (R_2)_n)$$

$$p_n(x) \approx 2(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots + \frac{x^{2n-1}}{2n - 1} + R_n)$$

$$R_n = (R_1)_n - (R_2)_n = (-1)^n \int_0^x \frac{t^n}{1 + t}\, dt - \int_0^x \frac{t^n}{1 - t}\, dt$$

$R_n = \int_0^x \frac{t^n(-2^t)}{1 - t^2}\, dt$ if n is even.

$R_n = \int_0^x \frac{-2t^n}{1 - t^2}\, dt$ if n is odd.

Teacher's Commentary

Chapter 10

SIMPLE DIFFERENTIAL EQUATIONS

Solutions Exercises 10-1a

1. (a) $s(t) = \frac{(t-1)^4}{4} - \frac{5}{4}$

(b) $s(t) = -e^{-t} + e + e^{-1}$

(c) $f(x) = \frac{1}{2} \arctan \frac{x}{2} - \frac{\pi}{8}$

(Substitute $u = \frac{x}{2}$ and use the formula $\int \frac{1}{1+u^2}\,du = \arctan u$ or substitute $x = 2 \tan t$ and obtain the result directly.)

(d) $f(x) = \frac{1}{4} \log_e(1 + 2x^2) + 1$ (substitute $u = 1 + 2x^2$)

(e) $g(x) = \frac{1}{5}(2x - 1)^{5/2} + 10$ (substitute $u = 2x - 1$)

(f) $h(r) = \frac{1}{2} e^{r^2} - \frac{1}{2} e$ (substitute $u = r^2$)

2. Substitute $v = -x$, $dv = -dx$ to obtain

$$\int_a^t u(x)dx = -\int_{-a}^{-t} u(-v)dv$$

$$\begin{aligned}
\therefore\ D\int_a^t u(x)dx &= D\left(-\int_{-a}^{-t} u(-v)dv\right) \\
&= -D\int_{-a}^{-t} u(-v)dv \\
&= -u(-(-t))D(-t) \qquad \text{(chain rule)} \\
&= u(t)
\end{aligned}$$

3. $f(x) = \frac{x^3}{3} - x^2 + 7x + \frac{34}{3}$

4. (a) $v(t) = gt - 2$

(b) $s(t) = \frac{gt^2}{2} - 2t + 3$

5. (a) $v(t) = -\frac{1}{2}\cos 2t - \frac{3}{2}$

$v(\frac{\pi}{4}) = -\frac{3}{2}$

Since $v(\frac{\pi}{4})$ is negative, we know that the particle is moving downward (if the movement is vertical) or to the left (if the movement is horizontal) at time $t = \frac{\pi}{4}$.

(b) $s(t) = -\frac{1}{4}\sin t - \frac{3}{2}t + 3$

6. (a) $y = \frac{2}{3}x^{3/2}$

(b) $y = \frac{2}{3}x^{3/2} - \frac{2}{3}$

7. $y = -\frac{1}{3}\cos 3x + c$

8. (a) $y = \sin x$

$y' = \cos x$

$y'' = -\sin x = -y$

(b) $y = a^x$

$y' = (\log_e a)a^x$

$y'' = (\log_e a)^2 a^x = (\log_e a)^2 y$

(c) y' has degree 2

y'' has degree 1

$y^{(3)}$ has degree 0 ($y = c$, c constant)

$y^{(4)} = 0$

(d) From (b) $yy'' - (y')^2 = a^x(\log_e a)^2 a^x - [(\log_e a)a^x]^2 = 0.$

(e) $y = e^x \cos x$

$y' = e^x(\cos x - \sin x)$

$y'' = -2e^x \sin x$

$y'' - 2y' + 2y = -2e^x \sin x - 2e^x(\cos x - \sin x) + 2e^x \cos x$

$= 0$

(f) $y = e^{x^2}$

$y' = 2xe^{x^2}$ (by the chain rule, with $u = x^2$)

$= 2xy$

9. Solving the initial value problem

$$s'(t) = (t - 1)e^{-t} \qquad s(0) = 1$$

we obtain

$$s(t) = 1 - te^{-t}.$$

We wish to show that $s(t) > 0$ for $t \geq 0$. We do this by determining the minimum value of $s(t)$, $t \geq 0$.
The minimum value could occur at $t = 0$, at points where $s'(t) = 0$, or at the limit as t becomes large:

(a) $s(0) = 1$

(b) $s'(t) = (t - 1)e^{-t} = 0$ only where $t = 1$

(c) as t becomes large, te^{-t} becomes small and $\lim\limits_{t \to 0} 1 - te^{-t} = 1$

Thus the minimum value of $s(t)$ is $s(1) = 1 - \frac{1}{e} > 0$.

10. Given $f'(x) = \frac{1}{1 + x}$ $f(0) = 1$, $f(1) = 10$.

$f(x)$ must have the form

$$f(x) = \int \frac{1}{1 + x}\,dx + c$$

$$= \log_e(1 + x) + c$$

The first condition gives $c = 1$, and the second condition gives $c = 10 - \log_e 2 \neq 1$. It is obviously impossible to pick c to satisfy both conditions.

11. $D(\log_e - x) = \frac{1}{-x}(-1) = \frac{1}{x}$ holds if $-x > 0$, that is, if $x < 0$.

12. Let $$f_1(x) = \begin{cases} \log_e(-x), & x < 0 \\ \log_e x, & x > 0 \end{cases}$$

$$f_2(x) = \begin{cases} c + \log_e(-x), & x < 0 \\ \log_e x, & x > 0 \end{cases} \qquad c \text{ any constant}$$

This does not contradict Theorem 10-1a, since $x \to \frac{1}{x}$ is not continuous at $x = 0$. Theorem 10-1a does imply that there is a unique solution to this problem for $x > 0$.

Solutions Exercises 10-1b

1. $y = 2e^{x^2/2} - 1$

Solution of this problem using separation of variables would proceed as follows:

$$\frac{dy}{dx} = xy + x = x(y + 1) = \frac{x}{(y + 1)^{-1}}$$

so $\int (y + 1)^{-1} dy = \int x\, dx + c$

$$\log_e(y + 1) = \frac{x^2}{2} + c$$

using $y(0) = 1$, we have $\log_e 2 = c$. Thus the solution is

$$\log_e(y + 1) = \frac{x^2}{2} + \log_e 2$$

and taking exponents of both sides gives the above result.

2. $\frac{x^2}{2} + y^2 = \frac{3}{2}$.

3. $y = 2e^x$

Using separation of variables, we would take

$$F(y) = y^{-1}, \quad G(x) = 1.$$

4. $y = \sec x$

Take $F(y) = y^{-1}$, $G(x) = \tan x$.

5. $y = -\sec x$

6. $y = \frac{-1}{x - \frac{21}{10}} = \frac{10}{21 - 10x}$

Take $F(y) = y^{-2}$, $G(x) = 1$.

7. $y = \tan(x + \log_e x - 1)$

Take $F(y) = \frac{1}{y^2 + 1}$, $G(x) = 1 + \frac{1}{x}$.

Recall that $\int \frac{1}{y^2 + 1} dy = \arctan y$ and that $\arctan 0 = 0$.

8. $(\log_e x)^2 - 2 \log_e y = (\log_e 2)^2 - 2 \log_e 2$

Take $F(x) = \dfrac{\log_e x}{x}$, $G(y) = \dfrac{1}{y}$ noting that $\dfrac{dx}{dy}$, not $\dfrac{dy}{dx}$, is given.

The value of $\displaystyle\int \frac{\log_e x}{x}\,dx$ can be determined by substituting $u = \log_e x$, $du = \dfrac{1}{x}\,dx$.

9. $s = t$

Take $F(s) = e^{-s}$, $F(t) = e^{-t}$ and the solution is $-e^{-s} = -e^{-t}$.

Solutions Exercises 10-2

1. $y = \frac{c_4x^4}{24} + \frac{c_3x^3}{6} + \frac{c_2x^2}{2} + c_1x + c_0$ where $c_i = y^{(i)}(0)$, $i = 0, \ldots, 4$

2. $f(x) = c_1x + 1$

3. If $f''' = 0$, then f has the form

$$f(x) = \alpha x^2 + \beta x + \gamma$$

so that $f'(x) = 2\alpha x + \beta$ and $f''(x) = 2\alpha$. Using the initial conditions, we solve the equations

$$f(0) = \alpha(0)^2 + \beta(0) + \gamma = a$$

$$f'(0) = 2\alpha(0) + \beta = b$$

$$f''(0) = 2\alpha = c$$

to obtain $\alpha = \frac{c}{2}$, $\beta = b$, $\gamma = a$ so that f must be

$$f(x) = \frac{cx^2}{2} + bx + a.$$

4. f has degree at most two.

5. (a) $f(x) = \frac{ax^3}{6} + \frac{bx^2}{2} + cx + d$

   (b) $f(x) = \frac{x^3}{6} + ax + b$

   (c) $f(x) = \frac{x^5}{60} + \frac{ax^2}{2} + bx + c$

6. $f^{(n+k+1)} = 0$, so from Theorem 10-2a, degree of $f \leq n + k$.

7. $f(x) = \frac{x^3}{3} + (b - \frac{1}{2})x + (a - b + \frac{1}{3})$

   The coefficients $b - \frac{1}{2}$ and $a - b + \frac{1}{3}$ are determined uniquely by the method of Exercise 3.

8. We are given

$$s'' = 32 \qquad s(0) = 0 \qquad s'(0) = 60.$$

Solving gives

$$s(t) = 16t^2 + 60t.$$

The ball strikes the ground when $s(t) = 2000$. Solving $16t^2 + 60t = 2000$ gives $t_1 = \frac{\sqrt{8225} - 15}{8} \approx 9\frac{1}{2}$ so that $s(t)$ is valid for $0 \leq t \leq t_1$.

9. If we let $s(t)$ = distance above the ground, then we are given

$$s'' = -32 \qquad s(0) = 2000 \qquad s'(0) = 64.$$

Solving gives

$$s(t) = -16t^2 + 64t + 2000.$$

This is valid until $s(t) = 0$. Solving $-16t^2 + 64t + 2000 = 0$ gives $t_1 = 2 + \sqrt{129} \approx 13\frac{1}{2}$, so that $s(t)$ is valid for $0 \le t \le t_1$.

10. For the velocity, we have

$$v - 2 = \int_0^t (3u - 2)du$$

so that

$$v = \frac{3}{2}t^2 - 2t + 2.$$

For the distance d covered in the first second we have

$$\begin{aligned} d &= \int_0^1 (\tfrac{3}{2}t^2 - 2t + 2)dt \\ &= \tfrac{1}{2} - 1 + 2 \\ &= \tfrac{3}{2} \text{ in centimeters.} \end{aligned}$$

11. The particle is at rest when $v = 0$ or $4t^2 - 14t + 6 = 2(2t - 1)(t - 3) = 0$; that is, when $t = \frac{1}{2}$ or $t = 3$. The displacement is

$$\begin{aligned} \int_{1/2}^3 (4t^2 - 14t + 6)dt &= \tfrac{4}{3}(27 - \tfrac{1}{8}) - 7(9 - \tfrac{1}{4}) + 6(3 - \tfrac{1}{2}) \\ &= \frac{-125}{12}. \end{aligned}$$

The sign indicates a displacement in the negative sense along the line. The distance covered is the absolute value of the displacement; that is, $\frac{125}{12}$.

12. In Equation (5) of the text we have $z_0 = 0$ (initial ground level) so that the height is given by

$$z = v_0 t - 16t^2.$$

The maximum height occurs when

$$\frac{dz}{dt} = v_0 - 32t$$

is zero. Thus, we ask that

$$Z_{max} = v_0\left(\frac{v_0}{32}\right) - 16\left(\frac{v_0}{32}\right)^2 \geq 100$$

or

$$\frac{v_0^2}{32} - \frac{1}{2}\,\frac{v_0^2}{32} = \frac{v_0^2}{64} \geq 100$$

or

$$v_0^2 \geq 6400.$$

Since v_0 is positive the condition is

$$v_0 \geq 80 \qquad \text{(in feet per second).}$$

13. Under the conditions of the problem take $z_0 = 200$ and $v_0 = -40$ in Equation (5):

$$\begin{aligned} z &= 200 - 40t - 16t^2 \\ &= -8(2t - 5)(t + 5). \end{aligned}$$

The time $t > 0$ when the stone reaches the ground $(z = 0)$ is $t = \frac{5}{2}$ (in seconds). At that time

$$v = -40 - 32t = -120 \qquad \text{(in feet per second).}$$

14. If the stone is dropped we have $v_0 = 0$, or

$$z = 200 - 16t^2.$$

The stone reaches the ground $z = 0$ when

$$t = \sqrt{\frac{200}{16}} = \frac{5\sqrt{2}}{2} \qquad \text{(in seconds)}$$

and then

$$v = -32\left(\frac{5\sqrt{2}}{2}\right) = -80\sqrt{2} \approx -113 \qquad \text{(in feet per second).}$$

15. Proceed as before, to obtain

$$z = 200 + 40t - 16t^2$$
$$= -8(t - 5)(2t + 5).$$

The stone reaches the ground $(z = 0)$ at $t = 5$ and the velocity is then

$$v = 40 - 32t = -120 \qquad \text{(in feet per second)}.$$

This result is consistent with the result of Number 13, since the stone returns to its initial elevation with the negative of its initial velocity.

16. (a) The acceleration of a freely falling body is

$$v' = \pm g$$

(where the sign depends on the direction of measurement), so that

$$v = \pm gt + c,$$

which is a linear function in t.

(b) If $s' = v = at + b$ then

$$s = \frac{at^2}{2} + bt + c.$$

17. We are given

$$s'' = 32 \qquad s(0) = 0 \qquad s'(0) = 0.$$

Solving gives

$$s(t) = 16t^2.$$

Although $s(t)$ does not depend on A, the interval of validity does depend on A, for the expression holds only for

$$0 \leq t \leq \frac{\sqrt{A}}{4}.$$

18. We are given

$$s'' = g \qquad s(0) = 0 \qquad s'(0) = 0.$$

So

$$v = s' = gt$$

which expresses the fact that the velocity is proportional to time.

Also
$$s(t) = \frac{gt^2}{2}$$

so that
$$s(2t) = 4\,\frac{gt^2}{2} = 4s(t).$$

19. $s_1'' = -32$, since the direction of measurement is opposite to the direction of fall.

$$s_1(0) = 2000$$

$$s_1'(0) = 0$$

$$s_1(t) = -16t^2 + 2000 = 2000 - s(t).$$

20. (a) If the air resistance is negligible, and $s(t)$ is the distance from the point of drop, the method of Exercises 8 and 9 tells us that the body strikes the ground when $t = 25$.

If it is known that the body does not strike the ground until $t = 30$, we can see that air resistance has a significant retarding effect.

(b) If $s(t)$ is the distance (measured downward) from the point of throw,

$$s'' = 32, \qquad s(0) = 0,$$

and we wish to determine $v(0) = s'(0)$ in order that

$$s(30) = 10{,}000.$$

Integrating twice gives

$$s(t) = 16t^2 + at + b$$

$s(0) = 0$ gives

$$s(t) = 16t^2 + at$$

and $s(30) = 10{,}000$ gives

$$a = -146\,\frac{2}{3}\,.$$

But $s'(0) = 32(0) + a = a$ so the ball must be thrown upward with an initial velocity of $146\frac{2}{3}$ feet per second.

21. (a) $y = f + g$

$y' = f' + g'$

$y'' = f'' + g'' = 0 + h = h$

(b) $y = f - g$

$y' = f' - g'$

$y'' = f'' - g'' = h - h = 0$

(c) If $y'' = h$, $f'' = h$, then (b) tells us that $g = y - f$ satisfies $g'' = 0$.

Hence, y has the form

$$y = f + (y - f) = f + g \quad \text{where} \quad g'' = 0,$$

and (a) tells us that such a y is a solution to $y'' = h$.

22. (a) Integrating $f''(x) = \sin x$ twice we obtain $f(x) = -\sin x$.

So $f(x) = -\sin x$ is a solution to $y'' = \sin x$.

The solutions to $y'' = 0$ are of the form $g(x) = ax + b$.

Applying 21(c) we see that all solutions to $y'' = \sin x$ are of the form $y(x) = -\sin x + ax + b$.

(b) $y(x) = \dfrac{4x^{5/2}}{15} + ax + b$

Solutions Exercises 10-3

1. We have $W(t) = W_0 e^{-ct}$ and wish to find

$$\frac{W(7.7)}{W(0)} = e^{-7.7c} \qquad \frac{W(30.8)}{W(0)} = e^{-30.8c}$$

Given that $\frac{K}{2} = Ke^{-3.85c}$ we have $e^{-3.85c} = \frac{1}{2}$

$$e^{-7.7c} = (e^{-3.85c})^2 = (\tfrac{1}{2})^2 = \frac{1}{4} .$$

After 7.7 days, $\frac{1}{4}$ is left.

$$e^{-30.8c} = (e^{-3.85c})^8 = (\tfrac{1}{2})^2 = \frac{1}{256} .$$

After 30.8 days, $\frac{1}{256}$ is left.

We can solve this problem more quickly if we notice that both 7.7 and 30.8 are integral multiples of the half-life $h = 3.85$.

After one half-life, half of the original material remains; after two half-lives, half of that half, or one quarter of that original material remains, and so on; so that if n is the number of half-lives,

$$W(nh) = W(0) \times (\tfrac{1}{2})^n$$

and $W(7.7) = W(2 \times 3.85) = W(0) \times (\frac{1}{2})^2 = W(0) \times \frac{1}{4}$

and $W(30.8) = W(8 \times 3.85) = W(0) \times (\frac{1}{2})^8 = W(0) \times \frac{1}{256}$.

2. After 13.4 minutes $\frac{\sqrt{2}}{2} \sim .707$ is left. After 80.4 minutes, $\frac{1}{8}$ is left.

Note that here we are dealing with one half of a half-life, and with three half-lives, so that the resulting fractions are

$$(\tfrac{1}{2})^{1/2} = \sqrt{\tfrac{1}{2}} = \frac{\sqrt{2}}{2} \quad \text{and} \quad (\tfrac{1}{2})^3 = \frac{1}{8} .$$

3. The half-life is 3.05 minutes.

Here $\frac{1}{16} = (\frac{1}{2})^4$ is the fraction remaining after 12.2 minutes. Hence, 12.2 minutes must be 4 half-lives, so that one half-life is 3.05 minutes.

4. We have $W(t) = Ke^{-ct}$ and are given that

$$\frac{49}{50} K = Ke^{-ct} \quad \text{and} \quad W(t_0) = 2.$$

The solution is

$$W(t + t_0) = 2(\frac{49}{50}).$$

5. We have $W(t) = Ke^{-ct}$ and are given that

$$\frac{3}{4} K = Ke^{-c\alpha} \qquad \alpha = 3.36 \times 10^4$$

and wish to find t so that $\frac{1}{2} = e^{-ct}$. The solution is

$$t = \frac{\log_e 2}{c} = \frac{\alpha \log_e 2}{\log_e 4 - \log_e 3} \sim 8.10 \times 10^4.$$

6. We have $W(t) = me^{-ct}$ and are given that

$$.277m = me^{-3000c},$$

$$W(0) = m = 2$$

$$\begin{aligned} W(810) &= 2e^{-810c} \\ &= 2(e^{-3000c})^{.27} \\ &= 2(.277)^{.27} \\ &\sim 2(10^{.44248})^{.27} \\ &= 2(10^{-.55752 \times .27}) \\ &\sim 10^{.30103} \times 10^{-.15053} \\ &= 10^{.15050} \\ &\sim \sqrt{2} \\ &\sim 1.4142 \end{aligned}$$

If we use 1620 years as the half-life of radium (Example 10-3b), then

$$W(810) = 2e^{-810c} = 2(e^{-1620c})^{1/2} = 2(\frac{1}{2})^{1/2} = \sqrt{2}.$$

7. The population in 1965 was 43,200.

8. If we use the assumptions of **Number 7 to** calculate the 1960 population from the population figures of 1920 and 1940, we obtain the estimate of 163.7 million. Since this is far below the actual population of 180 million, we must conclude that the rate of growth of U.S. population is not porportional to the population size, but somewhat greater.

9. (a)
$$\begin{aligned} f(x) &= f(x+0) \\ &= f(x)f(0) \qquad \text{since } f(x+y) = f(x)f(y) \text{ for } \underline{\text{all}}\ x, y. \\ &= f(x)\cdot 0 \\ &= 0 \end{aligned}$$

(b) $f(x) = f(x)f(0)$ implies that either $f(x) = 0$ or $f(0) = 1$. If $f(0) \neq 0$, then $f(x) \neq 0$ and $f(0) = 1$ must be true, or alternately,

$$\begin{aligned} f(0) &= f(0+0) \\ &= f(0)f(0) \\ &= [f(0)]^2 \end{aligned}$$

which holds only if $f(0) = 0$ or $f(0) = 1$.

(c) If $f(0) = 0$,

(i) gives $f(x) = 0$, and $f'(x) = f'(0)f(x) = 0$.

If $f(0) \neq 0$

(ii) gives $f(0) = 1$.

Since f is differentiable at 0, we then have

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{f(h) - 1}{h} \\ f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x)f(h) - f(x)}{h} \\ &= \lim_{h \to 0} f(x) \frac{f(h) - 1}{h} \\ &= f(x) \lim_{h \to 0} \frac{f(h) - 1}{h} \\ &= f(x)f'(0) \end{aligned}$$

(d) From (b) and (c) we have

$$f'(x) = cf(x) \qquad f(0) = 1 \qquad c = f'(0).$$

Theorem 10-3a tells us that

$$f(x) = Ke^{cx}.$$

Also $f(0) = K = 1$, so that

$$f(x) = e^{cx} \quad \text{where} \quad c = f'(0).$$

10. If $c < 0$,

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} Ke^{cx} = \begin{cases} \infty, & K > 0 \\ -\infty, & K < 0 \end{cases}$$

so that $|f(x)|$ becomes arbitrarily large as x becomes large.

If $c > 0$,

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} Ke^{cx} = \begin{matrix} \infty, & K > 0 \\ -\infty, & K < 0 \end{matrix}$$

so that $|f(x)|$ becomes arbitrarily large as x becomes large.

11. (a) $(f + g)' = f' + g'$

$$= cf + (cg + h)$$

$$= c(f + g) + h$$

(b) $(f - g) = f' - g'$

$$= (h + cf) - (h + cg)$$

$$= c(f - g)$$

(c) We are given $g' = cg + h$.
Suppose y is any solution to $y' = cy + h$. We can write $y = g + (y - g) = g + f$ and part (b) gives $(y - g)' = c(y - g)$.
Conversely, if $y = g + f$ where $f' = cf$, then part (a) gives $y' - cy = h$.

12. (a) $(x)' - x = 1 - x$

(b) 11(c) and 12(a) **state** that all solutions to $y' - y = 1 - x$ are of the form $y = x + f$ where $f' = cf$.
From Theorem **10-3a**, f must have the form

$$f(x) = Ke^{cx}.$$

13. (a) Show that $y = \frac{32}{k}$ is one solution to $y' = 32 - ky$, then apply 11(c) as was done in 12(b).

(b) Set $x = 0$, $y = 0$ in $y = \frac{32}{k} + Ke^{-kx}$, and $K = -\frac{32}{k}$ is determined uniquely.

14. The acceleration $v'(t)$ is the sum of

(i) the acceleration due to gravity

$$g = 32 \frac{\text{ft}}{\text{sec}^2},$$

(ii) the acceleration due to air resistance, which is proportional to v

$$r = kv \quad \text{where } k \text{ is constant.}$$

(a) Since gravity and air resistance are operating in opposite directions,

$$v' = 32 - kv \quad \text{where} \quad k > 0.$$

Since $v(0) = 0$, 13(b) gives

$$v(t) = \frac{32}{k}(1 - e^{-kt}).$$

(b) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$\sim 1 + x$ if $|x|$ is small.

Hence, if $|kt|$ is small

$$v(t) = \frac{32}{k}(1 - e^{-kt})$$

$$\sim \frac{32}{k}(1 - 1 + kt) = 32t.$$

(c) As $t \to \infty$, $e^{-kt} \to 0$ so $\lim\limits_{t \to \infty} v(t) = \frac{32}{k}$.

15. Newton's Law of Cooling states

$$T'(t) = c(T(t) - B)$$

or, since

$$T'(t) = (g(t) + B)' = g'(t)$$

it could be stated in the form

$$g'(t) = cg(t).$$

(a) The result comes directly from Theorem 10-3a.

(b) $A = g(0) = T(0) - B$

This is the initial difference in temperature between the body and the surrounding medium.

(c) If $T(0) > B$, then $A = g(0) > 0$.

16. From 15(a) and 15(c), we have

$$T(t) = 20 + 80e^{ct}.$$

(a) Using $T(10) = 80$ we find that $T(20) = 65$. Thus the temperature after 20 minutes is 65°C.

(b) We wish to solve the equation

$$T(t) = 40 \quad \text{for} \quad t.$$

$$t = \frac{10 \log_e \frac{1}{4}}{\log_e \frac{3}{4}}$$

$$= \frac{10 \log_e 4}{\log_e 4 - \log_e 3}$$

$$\sim 47.1$$

17. Case 1. Add the cream immediately. The temperature of the mixture is then

$$180\left(\frac{6}{7}\right) + 70\left(\frac{1}{7}\right) = \frac{660}{7} + 70$$

since $\frac{6}{7}$ of the mixture (coffee) is at 180° and $\frac{1}{7}$ (cream) is at 180° and $\frac{1}{7}$ (cream) is at 70°. If $T(t)$ is the temperature t minutes later then from Number 13.

$$T(t) = B + Ae^{ct}$$

where B = room temperature = 70

and $A + B$ = initial temperature = $\frac{660}{7} + 70.$

Hence, $A = \frac{660}{7}$ so $T(t) = \frac{660}{7} e^{ct} + 70.$

Case 2. Add the cream at time t. In this case, the temperature before the cream is added is

$$70 + (180 - 70)e^{ct} = 70 + 110e^{ct}$$

since 70 = room temperature

and 180 - 70 = initial temperature difference.

Thus, mixing at time t gives the temperature

$$(70 + 110e^{ct})\left(\frac{6}{7}\right) + 70\left(\frac{1}{7}\right) = \frac{660}{7} e^{ct} + 70,$$

which is the same as found in Case 1. Therefore, it makes no difference when the cream is added.

Solutions Exercises 10-4

1. The assumptions used in Example 10-4a are

   (i) Newton's Second Law of Motion:

$$F = ma \quad \text{where} \quad y'' = a.$$

   (ii) Hooke's Law:

$$F = -ky$$

   where k is the spring constant.

   That is

$$ky = my'' \quad \text{or} \quad y'' + \left(\sqrt{\frac{k}{m}}^2\right)y = 0.$$

   Solving, we obtain

$$y = A \sin \omega t + B \cos \omega t$$

   where

$$\omega = \sqrt{\frac{k}{m}} \qquad A = \frac{f'(0)}{\omega} \qquad B = f(0)$$

   (a) We are given $f(0) = -2$, $f'(0) = 0$ and so $f(t) = -2 \cos \omega t$.

   (b) We are given $f(0) = 0$, $f'(0) = 1$ and so $f(t) = \frac{1}{\omega} \sin \omega t$.

   (c) $f(0) = 0$, $f'(0) = -2$ $\quad \therefore \ f(t) = -\frac{2}{\omega} \sin \omega t$

   (d) $f(0) = -2$, $f'(0) = 3$ $\quad \therefore \ f(t) = \frac{3}{\omega} \sin \omega t - 2 \cos \omega t$

2. $f(0) = 2$, $f'(0) = 0$

   $f(t) = 2 \cos \omega t$

   (a) "g has a period of p" means $g(x) = g(x + p)$ for all x, where $p > 0$ is the smallest constant for which this is true.
   Now $\cos x = \cos(x + 2\pi)$, i.e., $\cos x$ has period 2π.

$$\begin{aligned} f(t) &= 2 \cos \omega t \\ &= 2 \cos (\omega t + 2\pi) \\ &= 2 \cos(\omega(t + \frac{2\pi}{\omega})) \\ &= f(t + \frac{2\pi}{\omega}) \end{aligned}$$

   $f(t)$ has a period of $\frac{2\pi}{\omega}$

(b) The mass is furthest right at $t = 0$, so it is furthest left after half a period, i.e., at $t_1 = \frac{\pi}{\omega}$.

$$\begin{aligned} f(t_1) &= 2 \cos \omega(\tfrac{\pi}{\omega}) \\ &= 2 \cos \pi \\ &= -2 \end{aligned}$$

The mass moves at most 2 units to the left. (This can be seen directly by noting that minimum $K \cos x = -(\text{maximum } K \cos x) = -K$.)

(c) We wish to know the smallest value $t = t_2 \geq 0$ for which

$$2 \cos \omega t = 0.$$

Solving, we obtain

$$t_2 = \frac{\pi}{2\omega}.$$

(d) $f'(t) = -2\omega \sin \omega t$ and the velocity when the mass first crosses 0 is given by

$$\begin{aligned} f'(t_2) &= -2\omega \sin \omega(\tfrac{\pi}{2\omega}) \\ &= -2\omega \end{aligned}$$

(e) Velocity when mass is furthest right is $f'(0) = 0$.
Velocity when mass is furthest left is given by: (from part (b))

$$\begin{aligned} f'(\tfrac{\pi}{\omega}) &= -2\omega \sin \omega(\tfrac{\pi}{2\omega}) \\ &= -2\omega(0) \\ &= 0 \end{aligned}$$

This is evident immediately if we recall that the maximum and minimum values of a function f exist only when $f'(x) = 0$ or $f'(x)$ is not defined; and that the derivative of the cosine function is everywhere defined.

Intuitively, the mass is changing direction at the extreme points of its path. That is to say the velocity is changing from negative to positive (or positive to negative), and must be zero.

3. (a) The position of a mass on a spring is given by

$$f(t) = \frac{f'(0)}{\omega} \sin \omega t + f(0) \cos \omega t.$$

$g(x) = \sin x$ and $h(x) = \cos x$ both have period 2π, i.e., $\sin x = \sin(x + 2\pi)$ and $\cos x = \cos(x + 2\pi)$.

$$\begin{aligned} f(t) &= \frac{f'(0)}{\omega} \sin \omega t + f(0) \cos \omega t \\ &= \frac{f'(0)}{\omega} \sin (\omega t + 2\pi) + f(0) \cos (\omega t + 2\pi) \\ &= \frac{f'(0)}{\omega} \sin[\omega(t + \frac{2\pi}{\omega})] + f(0) \cos \omega(t + \frac{2\pi}{\omega}) \\ &= f(t + \frac{2\pi}{\omega}) \end{aligned}$$

$\therefore$ The period of $f(t)$ is $\frac{2\pi}{\omega}$ and does not depend on $f(0)$ or $f'(0)$.

(b) Recall $\omega = \sqrt{\frac{k}{m}}$. So from (a) we have that the period of $f(t)$ is

$$p = \frac{2\pi\sqrt{m}}{\sqrt{k}}.$$

$\therefore$ If the mass is doubled, the period is multiplied by $\sqrt{2}$ and if the mass is quadrupled, the period is doubled.

4. (a) The approximation equation of motion is given by the solution of the differential equation

$$y'' + \omega y = 0 \qquad \omega = \sqrt{-\frac{k}{L}}$$

where k = acceleration due to gravity = -32

and L = length of the pendulum.

Thus, the equation of motion is

$$f(t) = \frac{f'(0)}{\omega} \sin \omega t + f(0) \cos \omega t$$

if $m = 10$, $L = 8$, $\theta(0) = \frac{\pi}{100}$, $f'(0) = 0$. Then

$$f(0) = L\theta(0) = \frac{2}{25}\pi$$

and

$$\begin{aligned} f(t) &= \frac{2}{25}\pi \cos(\sqrt{+\frac{32}{8}}\, t) \\ &= \frac{2}{25}\pi \cos 2t \end{aligned}$$

(b) The period of motion is π.

(c) No

(d) No

5. Theorem 10-4a states that all solutions $y = f(t)$ to the equation

$$y'' + \omega^2 y = 0 \qquad \omega \neq 0$$

are of the form

$$f(t) = A \sin \omega t + B \cos \omega t \tag{1}$$

but $\quad A \sin (\omega t + B) = (A \cos B) \sin \omega t + (A \sin B) \cos \omega t$

which is of the form (1).

6. Let y be any solution to the equation

$$y'' = -\omega^2 y + h. \tag{1}$$

Then y can be expressed in the form

$$y = (y - g) + g$$

where g is the solution to (1) given in the problem. Now

$$\begin{aligned}(y - g)'' &= y'' - g'' \\ &= -\omega^2 y + h + \omega^2 g - h \\ &= -\omega^2 (y - g).\end{aligned}$$

If we say $f = y - g$ then we have $y = f + g$ where $f'' = -\omega^2 f$.

7. (a) $y = x^2 \implies y'' + y = (2x)' + x^2 = 2 + x^2$

(b) $y'' + y = 2 + x^2$ is of the form $y'' + \omega^2 y = h$ where $\omega = 1$ and $h = 2 + x^2$. Thus, from Number 6, all solutions are of the form $y = f + g$ where g is any solution of $y'' + y = 2 + x^2$ and $f'' = -f$.

From Theorem 10-4a, all solutions to $f'' = -f$ are of the form

$$f = c_1 \sin x + c_2 \cos x$$

and from 7(a), $g = x^2$ is one solution of $y'' + y = 2 + x^2$. Therefore, all solutions to $y'' + y = 2 + x^2$ are of the form

$$y = x^2 + c_1 \sin x + c_2 \cos x.$$

8. (a) $y_1'' = (e^{cx})'' = (ce^{cx})' = c^2e^{cx} = c^2y_1$

$y_2'' = (e^{-cx})'' = (-ce^{-cx})' = c^2e^{-cx} = c^2y_2$

(b) $y'' = (c_1y_1 + c_2y_2)''$

$= c_1y_1'' + c_2y_2''$

$= c^2(c_1y_1' + c_2y_2')$

$= c^2y$

(c) $y(0) = c_1e^{c(0)} + c_2e^{-c(0)} = c_1 + c_2$

$y'(0) = c_1ce^{c(0)} - c_2ce^{-c(0)} = c(c_1 - c_2)$

$\frac{y'(0)}{c} = c_1 - c_2$

Solving for c_1, c_2, we obtain

$$c_1 = \frac{cy(0) + y'(0)}{2c}$$

and

$$c_2 = \frac{cy(0) - y'(0)}{2c} .$$

9. (a) By Theorem 10-3a, all solutions to $y' = cy$ have the form $y = Ke^{cx}$.

Now $$z = y' + cy$$

so $$z' = y'' + cy'.$$

But $y'' = c^2y$, so $z' = c^2y + cy' = cz$.

Therefore, by Theorem 10-3a, z has the form

$$z = c_1e^{cx}.$$

(b) From part (a)

$$z = y' + cy = c_1e^{cx}$$

is one solution to the equation $y'' = c^2y$. If we let y be any solution to $y'' = c^2y$ then (similar to Number 6) y may be expressed in the form

$$y = (y - Kz) + Kz.$$

Our hint suggests that we choose $K = \frac{1}{2c}$ in order to fascilitate the manipulation of constants.

Let $$Z = y - \frac{z}{2c} = y - \frac{c_1}{2c}e^{cx}.$$

Then
$$\begin{aligned} Z' &= y' - c\,\frac{c_1}{2c}e^{cx} \\ &= c_1e^{cx} - cy - c\,\frac{c_1}{2c}e^{cx} \qquad [\text{since } y' + cy = c_1e^{cx}] \\ &= -c\left(y - \frac{c_1}{2c}e^{cx}\right) \\ &= -cZ. \end{aligned}$$

As in part (a), Theorem 10-3a tells us that Z has the form

$$Z = k_2e^{-cx}.$$

So y has the form

$$\begin{aligned} y &= Z + Kz \\ &= k_2e^{-cx} + k_1e^{cx} \qquad\qquad k_1 = \frac{c_1}{2c} \end{aligned}$$

10. $y'' = y$, $y(0) = 1$, $y'(0) = 1$

By 9(b), y has the form

$$y = c_1e^{x} + c_2e^{-x} \qquad (1)$$

$$y(0) = c_1 + c_2 = 1$$

$$y'(0) = c_1 - c_2 = 1$$

$$c_1 = 1$$

$$c_2 = 0$$

$$y = e^{x}$$

This solution is unique because it is the only function of form (1) for which the boundary conditions are satisfied.

11. Note
$$z_1 + z_2 = e^{cx}$$

$$z_1 - z_2 = e^{-cx}$$

From 9(b), we have

$$
\begin{aligned}
y &= K_1 e^{cx} + K_2 e^{-cx} \\
&= K_1(z_1 + z_2) + K_2(z_1 - z_2) \\
&= (K_1 + K_2)z_1 + (K_1 - K_2)z_2 \\
&= c_1 z_1 + c_2 z_2 \\
y(0) &= c_1 \frac{e^{c0} + e^{-c0}}{2} + c_2 \frac{e^{c0} - e^{-c0}}{2} = c_1 \qquad c_1 = y(0) \\
y'(0) &= c_1 \frac{ce^{c0} - ce^{-c0}}{2} + c_2 \frac{ce^{c0} + e^{-c0}}{2} = cc_2 \qquad c_2 = \frac{y'(0)}{c}
\end{aligned}
$$

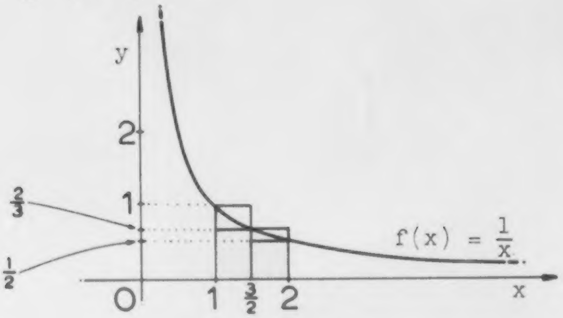
Solutions Exercises 10-5

1. $L(2) = \int_1^2 \frac{1}{x}\,dx$

Since $\frac{1}{x} > 0$ and strictly decreasing for $x > 0$, the maximum value of $\frac{1}{x}$ on any interval $0 < a \leq x \leq b$ will be $\frac{1}{a}$, and the minimum value will be $\frac{1}{b}$.

Choose the partition $x_0 = 1,\ x_1 = \frac{3}{2},\ x_2 = 2$.

L(2) is the measure of the shaded area



Then the upper sum will be

$$\frac{1}{1}\left(\frac{3}{2} - 1\right) + \frac{1}{\frac{3}{2}}\left(2 - \frac{3}{2}\right) = \frac{1}{2} + \frac{1}{3}\,.$$

And the lower sum will be

$$\frac{1}{\frac{3}{2}}\left(\frac{3}{2} - 1\right) + \frac{1}{2}\left(2 - \frac{3}{2}\right) = \frac{1}{3} + \frac{1}{4}\,.$$

Thus $$\frac{1}{3} + \frac{1}{4} < L(2) < \frac{1}{2} + \frac{1}{3}\,.$$

2. (a) Using the arguments of Number 1 for each positive integer k

$$\frac{1}{k} > \frac{1}{x} > \frac{1}{k+1} \quad \text{on} \quad k < x < k + 1.$$

And so

$$\left(\frac{1}{k+1}\right)(k + 1 - k) < \int_k^{k+1} \frac{1}{x}\, dx < \left(\frac{1}{k}\right)(k + 1 - k)$$

and

$$\sum_{k=1}^{n-1} \frac{1}{k+1} < \sum_{k=1}^{n-1} \int_k^{k+1} \frac{1}{x}\, dx = L(n) < \sum_{k=1}^{n-1} \frac{1}{k}\,.$$

Or, adding 1 to both sides, the left-hand inequality becomes

$$\sum_{k=1}^{n} \frac{1}{k} < 1 + L(n)$$

and adding $\frac{1}{n}$ to both sides of the right-hand inequality

$$\frac{1}{n} + L(N) < \sum_{k=1}^{n} \frac{1}{k}\,.$$

Combining, we get the required result.

(b) $\displaystyle\sum_{n=1}^{10^{100}} \frac{1}{n}$ lies between $L(10^{100}) - 1$ and $L(10^{100}) - \frac{1}{10^{100}}$, i.e.,

between $100L(10) - 1$ and approximately $100L(10)$.

Since $L(10) \sim 2.30258$ (from tables) $\displaystyle\sum_{n=1}^{10^{100}} \frac{1}{n}$ is about 230.

3. (a) On the interval $1 < x < a$

$$0 < \frac{1}{a} < \frac{1}{x} < 1$$

and $\displaystyle\int_1^a \frac{1}{a}\, dx < \int_1^a \frac{1}{x}\, dx < \int_1^a dx$

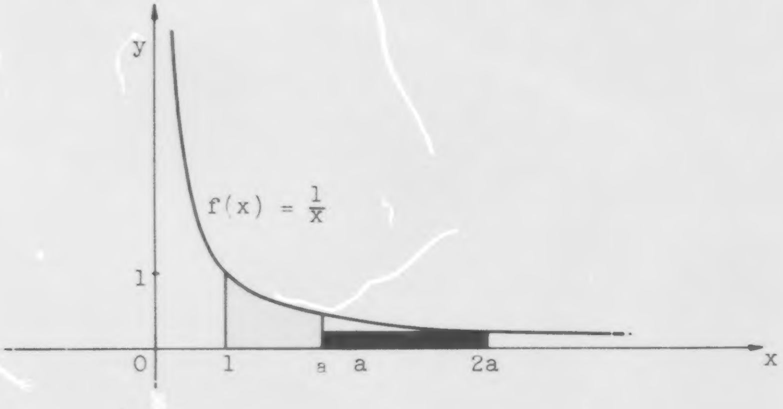
$$1 - \frac{1}{a} < L(a) < a - 1.$$

(b) $L(2a) = L(a) + \int_a^{2a} \frac{1}{x}\,dx$ $\qquad a > 1$

$$> L(a) + \int_a^{2a} \frac{1}{2a}\,dx$$

$$= L(a) + \frac{1}{2a}(2a - a)$$

$$= L(a) + \frac{1}{2}$$



$L(a)$ is the measure of the **light** shaded area

$\frac{1}{2}$ is the measure of the **dark** shaded area

(c) $L(a) = 2L(\sqrt{a})$

$< 2(\sqrt{a} - 1)$ $\qquad$ from 3(a)

$= 2\sqrt{a} - 2$

$< 2\sqrt{a}$

4. For all $x > 1$, $L(x) > 0$, so $\frac{L(x)}{x} > 0$ for any large x.

Also, from 3(c), for $x > 1$, $L(x) < 2\sqrt{x}$, so that $0 < \frac{L(x)}{x} < \frac{2}{\sqrt{x}}$.

Now $\lim_{x \to \infty} \frac{L(x)}{x} = 0$, $\lim_{x \to \infty} 0 = 0$ so by the Sandwich Theorem

$$\lim_{x \to \infty} \frac{L(x)}{x} = 0.$$

5. (a) $f(x) = L\left(\sqrt{\frac{x - 1}{x + 1}}\right)$

$$f'(x) = \sqrt{\frac{x + 1}{x - 1}}\frac{1}{2}\sqrt{\frac{x + 1}{x - 1}}\,\frac{(x + 1) - (x - 1)}{(x + 1)^2}$$

$$= \frac{1}{2}\,\frac{x + 1}{x - 1}\,\frac{2}{(x + 1)^2}$$

$$= \frac{1}{x^2 - 1}$$

(b) $f(x) = L(x\sqrt{1 - x})$

$$f'(x) = \frac{1}{x\sqrt{1 - x}}\left(\sqrt{1 - x} - \frac{x}{2\sqrt{1 - x}}\right)$$

$$= \frac{2(1 - x) - x}{2x(1 - x)}$$

$$= \frac{2 - 3x}{2x(1 - x)}$$

(c) $f(x) = L(L(x))$

$$f'(x) = \frac{1}{xL(x)}$$

6. $f(x) = xL(x)$

$f'(x) = L(x) + 1$

$f''(x) = \frac{1}{x}$

All of these functions are defined and continuous for $x > 0$.

$f''(x) > 0$ everywhere, so $f'(x)$ is strictly increasing, and $f(x)$ is concave.

Thus, $f(x)$ will have one minimum point where $f'(x) = 0$. That is, where

$$L(x) = -1$$

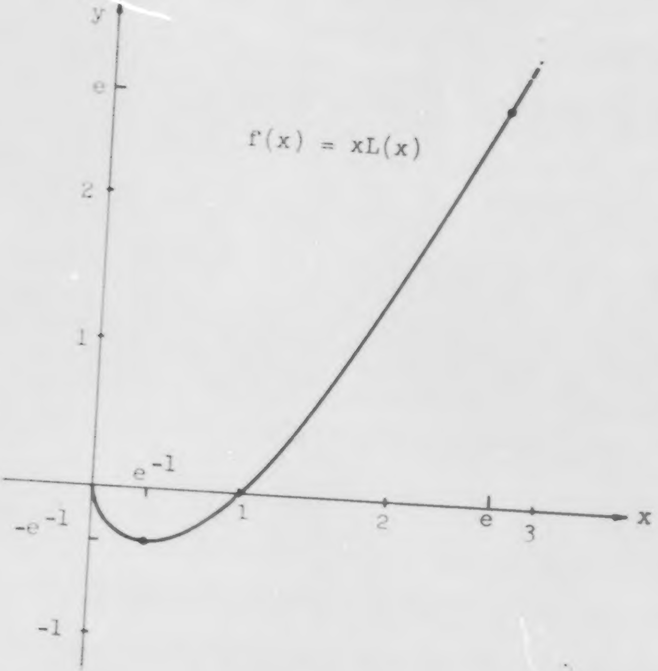
$$x = e^{-1}.$$

At this point, $f(e^{-1}) = -e^{-1}$.

As x becomes large, $L(x)$ becomes large, and $xL(x)$ becomes large.

On the interval $0 < x < 1$, $x > 0$ and $L(x) < 0$, so that $xL(x) < 0$.

Also, $f(1) = 0$, $f(e) = e$.



7. $a^x = E(xL(a))$

(a) $f(x) = (1 - x)^x = E(xL(1 - x))$

$$f'(x) = E(xL(1 - x))\left(L(1 - x) - \frac{x}{1 - x}\right)$$
$$= (1 - x)^x \frac{(1 - x)L(1 - x) - x}{1 - x}$$
$$= (1 - x)^{x-1}[(1 - x)L(1 - x) - x]$$

(b) $f(x) = (L(x))^x = E(xL(L(x)))$

$$f'(x) = E(xL(L(x)))\left(L(L(x)) + \frac{1}{xL(x)}\right)$$
$$= (L(x))^x \frac{xL(x)L(L(x)) + 1}{xL(x)}$$
$$= (L(x))^{x-1} \frac{xL(x)L(L(x)) + 1}{x}$$

(c) $f(x) = x^{1/x} = E(\frac{1}{x} L(x))$

$$f'(x) = E\left(\frac{1}{x} L(x)\right)\left(- \frac{L(x)}{x^2} + \frac{1}{x^2}\right)$$
$$= x^{\frac{1}{x} - 2}(1 - L(x))$$

8. $f(x) = x^x$ is defined for $x > 0$.

$f'(x) = x^x(L(x) + 1)$

$f''(x) = x^x(L(x) + 1)^2 + x^x \frac{1}{x}$

$f'(x)$ is everywhere increasing. So $f(x)$ is concave, and $f(x)$, has one minimum point where $f'(x) = 0$. That is where $L(x) = -1$, $x = e^{-1}$.

$$f(e^{-1}) = e^{-e^{-1}} \sim \frac{2}{3}$$

9. Let y be any solution of

$$y' = cy.$$

Let $z = E(-cx)y$. Then $z' = -cE(-cx)y + E(-cx)y'$

$$= E(-cx)(y' - cy) = 0.$$

Thus z is some constant K and $K = E(-cx)y$. Thus $y = K \cdot E(cx)$.

10. $f(x) = L(x) - 1$

$f'(x) = \frac{1}{x}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - x_n(L(x_n) - 1)$$

$$= x_n(2 - L(x_n))$$

$x_2 = x_1(2 - L(x_1))$

$= 2(2 - .7)$

$= 2.6$ $\qquad\qquad L(2.6) \simeq .96$

$x_3 = 2.6(2 - .96)$

$\simeq 2.7$

11. (a) $L'(1) = 1 = \lim\limits_{h \to 0} \frac{L(1 + h) - L(1)}{h} = \lim\limits_{h \to 0} L((1 + h)^{1/h})$

Now, since $L(x)$ is continuous

$$\lim_{h \to 0} L((1 + h)^{1/h}) = L(\lim_{h \to 0} (1 + h)^{1/h})$$

that is

$$L(\lim_{h \to 0} (1 + h)^{1/h}) = 1$$

$$E[L(\lim_{h \to 0} (1 + h)^{1/h})] = E(1)$$

$$\lim_{h \to 0} (1 + h)^{1/h} = e$$

(b) From 11(a), take $h = \frac{1}{n}$

$$e = \lim_{\frac{1}{n} \to 0} (1 + \frac{1}{n})^n \qquad \text{since } h = 0,\ n \neq 0 \text{ as } h = \frac{1}{n} \to 0$$

$$= \lim_{n \to \pm \infty} (1 + \frac{1}{n})^n$$

(c) Repeating the procedure of 11(a), we obtain

$$L'(\frac{1}{a}) = \lim_{h \to 0} \frac{L(\frac{1}{a} + h) - L(\frac{1}{a})}{h}$$

$$a = \lim_{h \to 0} L(1 + ah)^{1/h}$$

$$\lim_{h \to 0} (1 + ah)^{1/h} = e^a$$

and, as in 11(b), we obtain

$$\lim_{n \to \infty} (1 + \frac{a}{n})^n = e^a.$$

12. $A(x) = \int_0^x \frac{1}{\sqrt{1 - t^2}}\,dt, \qquad |x| < 1$

$$A'(x) = \frac{1}{\sqrt{1 - x^2}} = \frac{1}{y}, \qquad 0 < y \leq 1$$

$$= z, \qquad z \geq 1 > 0$$

Since $A'(x)$ is everywhere positive, $A(x)$ is strictly increasing. Also $A'(x)$ exists everywhere, so $A(x)$ is continuous.

13. (a) $A(0) = \int_0^0 \frac{1}{\sqrt{1 - t^2}}\,dt = 0 \qquad \therefore\ S(0) = 0$

(b) $S' = \frac{1}{A'(S)} = \sqrt{1 - S^2}$

(c) $S'(0) = \sqrt{1 - (S(0))^2} = 1$

(d) $S'' = \frac{-2SS'}{2\sqrt{1 - S^2}} = \frac{-S\sqrt{1 - S^2}}{\sqrt{1 - S^2}} = -S$

14. (a) $S'' + S = 0$ from 13(d)

$S''' + S' = 0$

i.e., $C'' + C = 0$

(b) $C' = S'' = -S$ from 13(d)

(c) $C(0) = S'(0) = 1$ from 13(c)

$C'(0) = -S(0) = 0$ from 13(a)

(d) $C^2 = (S')^2 = 1 + S^2$ from 13(b)

15. $y = S(x)$ is a solution of $y'' + y = 0$, $y(0) = 1$, $y'(0) = 0$.
Now any other solution to $y'' + y = 0$ is of the form

$$y(x) = z(x) + S(x) \qquad z(0) = 0 \qquad z'(0) = 0$$

z is a solution to $y'' + y = 0$.

$$\begin{aligned} D((z')^2 + z^2) &= 2z'z'' + 2zz' \\ &= 2z'(z'' + z) \\ &= 0 \end{aligned}$$

$$D((z')^2 + z^2) = 0 \qquad (z')^2 + z^2 = c$$

In particular, for $x = 0$

$$(z'(0))^2 + (z(0))^2 = 0 + 0 = c$$

so, for all real x

$$(z')^2 + z^2 = 0$$

which is true only if $z' = z = 0$.

16. $y = S(x)$ is a unique solution to the problem

$$y'' + y = 0 \qquad y(0) = 0 \qquad y'(0) = 1$$

We must show that

$$z(a + b) = S(a)C(b) + S(b)C(a)$$

satisfies these conditions.

Take any value of b and hold it constant. Then we can consider

$$S(a + b) = S_1(a)$$

and

$$z(a + b) = S(a)C(b) + S(b)C(a)$$

as functions of a.

Our conditions become:

$$y'' + y = 0 \qquad y(-b) = 0 \qquad y'(-b) = 1.$$

Now

$$\begin{aligned} z'(a) &= S'(a)C(b) + S(b)\,C'(a) \\ &= C(a)C(b) - S(b)S(a) \quad \text{from the definition of } C = S' \text{ and } 14(b). \\ z''(a) &= S''(a)C(b) + S(b)C''(a) \\ &= -S(a)C(b) - S(b)C(a) \quad \text{from } 13(d) \text{ and } 14(a) \\ &= -z(a). \end{aligned}$$

This shows that z satisfies $y'' - y = 0$.
Also

$$\begin{aligned} z(-b) &= S(-b)C(b) + S(b)C(-b) \\ &= -S(b)C(b) + S(b)C(b) \quad \ldots? \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} z'(-b) &= C(-b)C(b) - S(b)S(-b) \\ &= [C(b)]^2 + [S(b)]^2 \quad \ldots? \\ &= 1 \qquad\qquad \text{from } 14(d). \end{aligned}$$

These relations hold if the familiar properties

$$\sin(-x) = -\sin x$$

and

$$\cos(-x) = \cos x$$

hold for $S(x)$ and $C(x)$. We will show this is true.
Recall $S(x) = y \Leftrightarrow A(y) = x$.
To show $S(-x) = -y$, it is sufficient to show $A(-y) = -x$.

$$A(-y) = \int_0^{-y} \frac{1}{\sqrt{1 - t^2}}\,dt \qquad \text{substitute } t = -u,\ dt = -du$$

$$= -\int_0^{y} \frac{1}{\sqrt{1 - u^2}}\,du$$

$$= -A(y)$$

$$= -x$$

Thus $S(-x) = -S(x)$.

Taking derivatives of both sides, we get

$$D[S(-x)] = -C(x)$$

but

$$D[S(-x)] = S'(-x)D(-x)$$

$$= -C(-x)$$

and, substituting $-C(-x)$ for $D[S(-x)]$, we obtain

$$C(-x) = -C(x).$$

Teacher's Commentary

Appendix 5

AREA AND INTEGRAL

Solutions Exercises A5-1

1. Prove from Property 3 that if a region R is the union of n non-overlapping regions then

$$\alpha(R) = \alpha(R_1) + \alpha(R_2) + \ldots + \alpha(R_n).$$

We have

$$\begin{aligned}\alpha(R_1 \cup R_2 \cup R_3 \cup \ldots \cup R_n) &= \alpha(R_1) + \alpha(R_2 \cup R_3 \cup \ldots \cup R_n) \\ &= \alpha(R_1) + \alpha(R_2) + \alpha(R_3 \cup \ldots \cup R_n) \\ &= \ldots \\ &= \alpha(R_1) + \alpha(R_2) + \alpha(R_3) + \ldots + \alpha(R_n).\end{aligned}$$

The argument may be formalized by the use of mathematical induction.

2. Show that Property 2 is actually a consequence of Property 3 given that area is nonnegative. Incorporate the notion of complementary regions.

Let S be a subregion of T and let R be the complementary region of S in T; i.e., R is the region which does not overlap S and for which $R \,\hat{\cup}\, S = T$. (We deliberately omit the question of existence of R.) Since $\alpha(R) \geq 0$ and

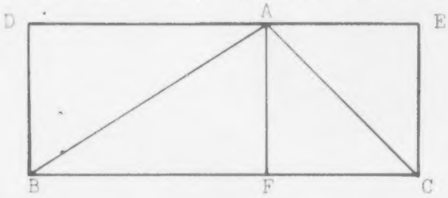
$$\alpha(R) + \alpha(S) = \alpha(T)$$

we have the desired result

$$\alpha(S) = \alpha(T) - \alpha(R) \leq \alpha(T).$$

3. (a) Using the given properties of area obtain the area of a triangle by elementary geometrical arguments.

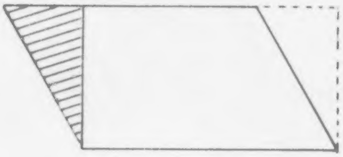
Let $\triangle ABC$ be the triangle and let BC be the longest side. We inscribe $\triangle ABC$ in a rectangle with one side on BC and suppose the foot of the perpendicular from A to BC lies on BC at F. (See Figure.)



From elementary geometry we have $\triangle AFC$ congruent to $\triangle CEA$, and $\triangle AFB$ to $\triangle BDA$. It follows that the area of the triangle ABC is half the area of the rectangle BCED and hence equal to half the product of the base BC and the altitude AF.

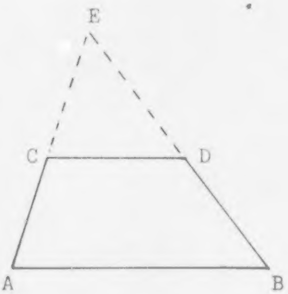
In this "proof" it is assumed from elementary geometry that congruent regions have equal area.

(b) Do the same for a trapezoid.



(i) If the sides of the trapezoid are parallel, the trapezoid is a parallelogram and has the same area as a rectangle of the same base and altitude.

(ii) If the sides are not parallel, extend them until they meet. The area is then the difference between the area of $\triangle EAB$ and $\triangle ECD$. In either case we get the usual formula.



4. If Property 4 is replaced by

Property 4^*: The area of a unit square is one,
Property 5 : Congruent regions have the same area,

show that the area of a square whose side is of length a is a^2.

The proof is given first for rational a then for arbitrary real a. If $a = \frac{1}{n}$ for a natural number n, then from the observation that the unit square can be subdivided into n^2 squares of sidelength $\frac{1}{n}$ it follows from Property 3 and new Properties 4^* and 5 that the area of the square is $\frac{1}{n^2}$. If $a = \frac{m}{n}$, then the square may be subdivided into m^2 congruent squares of sidelength $\frac{1}{n}$ and from the preceding result, the area is $\frac{m^2}{n^2}$. If now a is any real number, take $m = [\![an]\!]$. Then $m \leq an < m + 1$. It follows that the given square contains a square of sidelength $\frac{m}{n}$ and is contained in a square of sidelength $\frac{m+1}{n}$. Consequently, for the area A of the given square, by Property 2,

$$\frac{m^2}{n^2} \leq A < \frac{(m + 1)^2}{n^2} .$$

Consequently, for all natural numbers n,

$$\frac{(an - 1)^2}{n^2} < A < \frac{(an + 1)^2}{n^2} ,$$

or

$$a^2 - \frac{2a}{n} + \frac{1}{n^2} < A < a^2 + \frac{2a}{n} + \frac{1}{n^2} ;$$

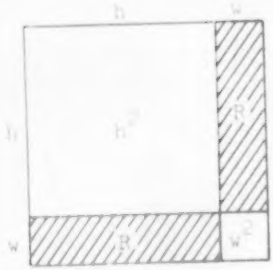
hence

$$|A - a^2| < \frac{2a + 1}{n} ,$$

from which the result follows.

5. Using Number 4, show that the area of a rectangle of height h and width w is hw.

Form the square of side $h + w$. If A is the area of the given rectangle, then from the figure and from Number 4 $(h + w)^2 = 2R + h^2 + w^2$. It follows that $R = hw$.



Solutions Exercises A5-2

1. (Requires Section A3-2(ii) for parts (d) and (e).) Use the summation method to find the area of the standard region defined by
(a) $f : x \to c, \quad 0 \le x \le b, \quad c > 0.$

Use a subdivision of the interval into n equal parts. Define $\alpha(S)$, $\alpha(T)$, and $\alpha(R)$ as in the text, using the respective minimum and maximum values of f in each subinterval. In this case the maximum on any interval is equal to the minimum, so that

$$\alpha(S) = \alpha(T) = \sum_{k=1}^{n} c \cdot \frac{b}{n} = bc.$$

(b) $f : x \to cx, \quad 0 \le x \le b, \quad c > 0.$

Here

$$\alpha(S) = \sum_{k=1}^{n} c \cdot \frac{(k-1)b}{n} \cdot \frac{b}{n} = \frac{cb^2}{n^2} \sum_{k=1}^{n} (k-1)$$

$$= \frac{cb^2}{n^2} \frac{n(n-1)}{2},$$

and

$$\alpha(T) = \sum_{k=1}^{n} c \cdot \frac{kb}{n} \cdot \frac{b}{n} = \frac{cb^2}{n^2} \sum_{k=1}^{n} k = \frac{cb^2}{n^2} \frac{n(n+1)}{2}.$$

From $\alpha(S) \le \alpha(R) \le \alpha(T)$ it follows that

$$-\frac{cb^2}{2n} \le \alpha(R) - \frac{cb^2}{2} \le \frac{cb^2}{2n},$$

for each natural number n. Consequently,

$$\alpha(R) = \frac{cb^2}{2}.$$

(c) $f : x \to x^2 + 2x, \quad 0 \leq x \leq b.$

Here

$$\alpha(T) = \sum_{k=1}^{n} \left[\left(\frac{kb}{n}\right)^2 + 2\,\frac{kb}{n} \right] \frac{b}{n}$$

$$= \frac{b^3}{n^3} \sum_{k=1}^{n} k^2 + \frac{2b^2}{n^2} \sum_{k=1}^{n} k$$

$$= \frac{b^3}{n^3} \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) + \frac{2b^2}{n^2}\, \frac{n(n+1)}{2}$$

$$= \frac{b^3}{3} + b^2 + \frac{b^2}{n} + \frac{b^3}{2n} + \frac{b^3}{6n^2}$$

and

$$\alpha(S) = \alpha(T) - \frac{b^3}{n} - \frac{2b^2}{n} .$$

As in Part (b), from $\alpha(S) \leq \alpha(R) \leq \alpha(T)$ obtain

$$\alpha(R) = \frac{b^3}{3} + b^2 .$$

(d) $f : x \to \sin(ax + b), \quad 0 \leq x \leq c;$ a, b, c such that $\sin(ax + b) \geq 0$ on $[0,\bar{c}]$.

The interval $[0,c]$ may be subdivided into two subintervals where f is strongly monotone. For simplicity, assume f is increasing on $[0,c]$. Then

$$\alpha(T) = \sum_{k=1}^{n} \frac{c}{n} \sin\left(\frac{akc}{n} + b\right).$$

In A3-2(ii), Equation (6), take $\frac{ac}{n}$ for a and $b + \frac{ac}{2n} - \frac{\pi}{2}$ for b.

Then

$$\alpha(T) = \frac{c}{n}\sum_{k=1}^{n} \sin\left(\frac{akc}{n} + b\right) = \frac{c \sin\left(\frac{ac}{2} + b + \frac{ac}{2n}\right)\sin\frac{ac}{2}}{n \sin\frac{ac}{2n}} .$$

Consequently, from the continuity of cos and $\lim\limits_{x\sim 0}\frac{\sin x}{x}$,

$$\lim_{n\sim\infty} \alpha(T) = \frac{2}{a}\sin\left(\frac{ac}{2} + b\right)\sin\frac{ac}{2} = \frac{\cos(ac + b) - \cos b}{a} .$$

A similar argument yields the same limit for $\alpha(S)$. By the Squeeze Theorem, it follows that $\alpha(R)$ is this common limit.

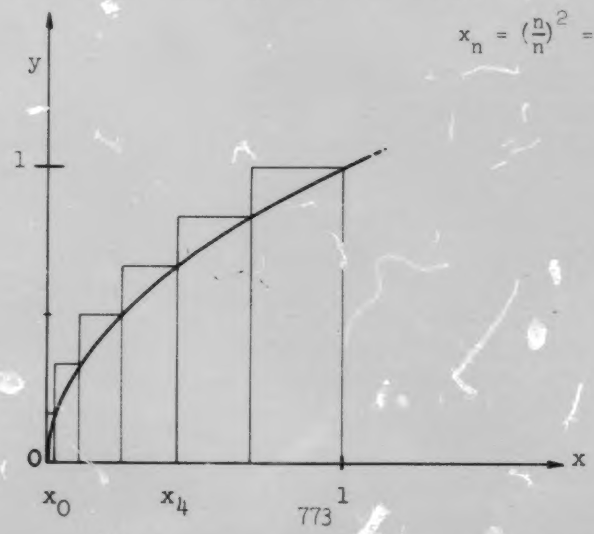
(e) $f : x \to \cos^2 x, \quad 0 \le x \le c$.
Use $\cos^2 x = \frac{\cos 2x + 1}{2}$. Proceed as in Part (d) with $\cos 2x = \sin(2x + \frac{\pi}{2})$.

$$\alpha(R) = \frac{\sin 2c}{4} + \frac{c}{2} .$$

2. Determine the area of the standard region $f : x \to \sqrt{x}$ on $[0,1]$. (The summation encountered will be similar to the one encountered in this section.)

In order to avoid a sum which involves square roots of natural numbers, it is most convenient to subdivide the interval $[0,1]$ in the following way:

$$x_0 = 0,\ x_1 = \left(\frac{1}{n}\right)^2,\ x_2 = \left(\frac{2}{n}\right)^2,\ \ldots,\ x_{n-1} = \left(\frac{n-1}{n}\right)^2,$$
$$x_n = \left(\frac{n}{n}\right)^2 = 1$$



For this subdivision

$$f(x_k) = \sqrt{\left(\frac{k}{n}\right)^2} = \frac{k}{n} .$$

Since $f : x \to \sqrt{x}$ is increasing on $[0,1]$

$$f(x_{k-1}) \leq f(x) \leq f(x_k) \quad \text{on} \quad [x_{k-1}, x_k],$$

and the upper sum has the form

$$\begin{aligned}
\alpha(T) &= \sum_{k=1}^{n} f(x_k)[x_k - x_{k-1}] \\
&= \sum_{k=1}^{n} \frac{k}{n}\left[\left(\frac{k}{n}\right)^2 - \left(\frac{k-1}{n}\right)^2\right] \\
&= \frac{1}{n^3} \sum_{k=1}^{n} k(2k - 1) \\
&= \frac{1}{n^3}\left[2\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k\right] \\
&= \frac{1}{n^3}\left[\frac{2n^3}{3} + n^2 + \frac{n}{3} - \frac{n(n+1)}{2}\right] \\
&= \frac{2}{3} + \frac{1}{2n} - \frac{1}{6n^2} .
\end{aligned}$$

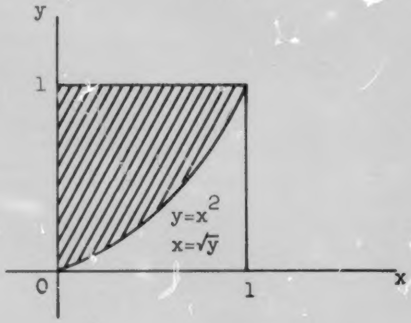
We have also

$$\alpha(S) = \alpha(T) - \frac{1}{n} = \frac{2}{3} - \frac{1}{2n} - \frac{1}{6n^2} .$$

It follows that the area is $\frac{2}{3}$.

3. Obtain the result of Exercise 2 using only the fact that the area under $f : x \to x^2$ on $[0,1]$ is $\frac{1}{3}$, together with the basic properties of area, without resort to summation techniques.

By Property 3, the area of the standard region under the graph of $x = \sqrt{y}$ on $[0,1]$ (the shaded region) plus the area of the standard region under the graph $y = x^2$ on $[0,1]$ (unshaded region) is 1.

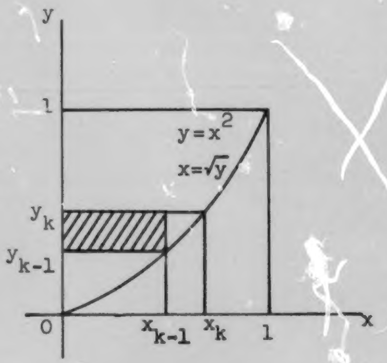


4. Show how the upper estimating sums for $\sqrt{x}$ are related term-by-term to the lower estimating sums for x^2. (Hint: Sketch a graph of $y = x^2$. Use this graph and the y-axis to represent the standard region defined by $\sqrt{x}$.)

Observe from the figure that a term from the upper sum (area of the unshaded rectangle) for $f : x \to x^2$ of the form $x_k^2(x_k - x_{k-1})$, corresponds to a term of the lower sum (area of the shaded rectangle) for $g : y \to \sqrt{y}$ of the form

$$\sqrt{y_{k-1}}\,(y_k - y_{k-1})$$

where $y_k = x_k^2$. Furthermore, the sum of the two is $x_k y_k - x_{k-1}y_{k-1}$. Adding the upper sum for f to the lower sum for g, we have



$$\begin{aligned}(x_1y_1 - x_0y_0) + (x_2y_2 - x_1y_1) + \dots + (x_ny_n - x_{n-1}y_{n-1}) \\ = x_ny_n - x_0y_0 \\ = 1.\end{aligned}$$

5. If $S_n = \sqrt{1} + \sqrt{2} + \ldots + \sqrt{n}$, show that

$$\frac{2}{3}\sqrt{n^3} < S_n < \frac{2}{3}\sqrt{n^3} + \sqrt{n}\,.$$

Divide $[0,1]$ into n equal subintervals. An upper bound for the area $\alpha(R)$ of the standard region $f : x \to \sqrt{x}$ on $[0,1]$ is

$$\alpha(T) = \sum_{k=1}^{n} \sqrt{\frac{k}{n}} \cdot \frac{1}{n} = \frac{1}{\sqrt{n^3}}\, S_n;$$

a lower bound is

$$\alpha(S) = \sum_{k=1}^{n} \sqrt{\frac{k-1}{n}} \cdot \frac{1}{n} = \frac{1}{\sqrt{n^3}}\,(S_n - \sqrt{n}).$$

Since

$$\alpha(S) \leq \alpha(R) \leq \alpha(T)$$

by Property 2, and $\alpha(R) = \frac{2}{3}$ by Number 2, the result follows immediately.

TC. A5-3. Integration by Summation Techniques

Once the student learns the Fundamental Theorem he may come to believe that the original conception of integral as the limit of a sum is not useful for analysis or computation. In this section it is shown that the formal integrals of polynomials and of the circular functions sin and cos can be obtained directly from the definition by summation techniques. This is something of a tour-de-force, but many students find the approach illuminating. Of course, summation remains valuable as a method of getting numerical estimates.

Solutions Exercises A5-3

1. Show simply, without repeating the argument of the text, that the lower sum over σ, $L = \sum_{k=1}^{n} x_{k-1}^{r}(x_{k-1} - x_k)$, also has the limit (7).

Since $L = U - h^{r+1} n^{r} = U - h(b - a)^{r}$, $\lim\limits_{h \to 0} L = \lim\limits_{h \to 0} U = \frac{a^{r+1}}{r + 1}$.

2. Employ Equation (8) of Section A3-2(ii) to obtain $\int_0^a \sin x \, dx$ for $0 < a \leq \frac{\pi}{2}$.

Replace a by $h = \frac{a}{n}$ in Equation (8) of Section A3-2(ii), and note that

$$\sin\frac{(n + 1)h}{2} \sin\frac{nh}{2} = \cos\frac{h}{2} - \cos\left(n + \frac{1}{2}\right)h$$

to obtain

$$\sum_{k=1}^{n} \sin kh = \frac{\cos\frac{h}{2} - \cos\left(n + \frac{1}{2}\right)h}{2 \sin\frac{h}{2}} .$$

Since $\sin x$ is increasing on $[0,a]$ the upper and lower sums for a subdivision of the interval into n equal parts are given by

$$U = \sum_{k=1}^{n} h \sin kh \quad \text{and} \quad L = \sum_{k=1}^{n} h \sin (k - 1)h.$$

Note that $L = U - h \sin a$. Insert $n = ah$ in the formula for $\sum_{k=1}^{n} \sin kh$ to obtain for the upper sum,

$$U = \frac{\frac{1}{2} h}{\sin \frac{1}{2} h} \cdot \left[\cos \frac{1}{2} h - \cos \left(a + \frac{1}{2} h\right)\right].$$

Use the continuity of the cosine and

$$\lim_{h \sim 0} \frac{\frac{1}{2} h}{\sin \frac{1}{2} h} = 1$$

to obtain

$$\lim_{h \sim 0} U = 1 - \cos a = \lim_{h \sim 0} L;$$

whence

$$\int_0^a \sin x \, dx = 1 - \cos a.$$

Solutions Exercises A5-4

1. By using upper and lower sum estimates evaluate the integral of each function f over the indicated interval.

   (a) $f(x) = 2 - x^2 \qquad 0 \leq x \leq 1$

   (b) $f(x) = x \qquad 1 \leq x \leq 2.5$

   (c) $f(x) = \frac{5}{2} \qquad 2.5 \leq x \leq 3$

   (d) $f(x) = 5 - x \qquad 3 \leq x \leq 5$

In each of the following we use subdivision into n equal parts.

(a) Since f is monotone decreasing, we take

$$L = \sum_{k=1}^{n} (2 - x_k^2)(x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} \left(2 - \frac{k^2}{n^2}\right)\frac{1}{n} \qquad \left(x_k = \frac{k}{n}\right)$$

$$= \frac{1}{n}\sum_{k=1}^{n} 2 - \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

$$= 2 - \frac{1}{3} - \frac{1}{2n} - \frac{1}{6n^2} .$$

Since $U = L + \frac{1}{n} - \frac{5}{3} + \frac{1}{2n} - \frac{1}{6n^2}$, the integral is $\frac{5}{3}$.

(b) Take $$U = \sum_{k=1}^{n} (1 + kh)h, \qquad (x_k = 1 + kh)$$

$$L = U(\sigma) - 1.5h$$

where $$h = \frac{(2.5 - 1)}{n} = \frac{1.5}{n}$$

$$U = \sum_{k=1}^{n} h + h^2 \sum_{k=1}^{n} k$$

$$= nh + h^2 \frac{n(n + 1)}{2} .$$

Then, since $nh = 1.5$,

$$U = 1.5 + \frac{(1.5)^2(1 + \frac{1}{n})}{2} .$$

It follows that the integral is

$$1.5 + \frac{2.25}{2} = 2.625.$$

(c) We take for both upper and lower sums

$$U = L = \sum_{k=1}^{n} \frac{5}{2} h$$

where $h = \frac{(3 - 2.5)}{n} = \frac{0.5}{n}$.

$$\sum_{k=1}^{n} \frac{5}{2} h = \frac{5nh}{2} = \frac{5}{2}(0.5) = 1.25.$$

(d) We take for the lower sum

$$L = \sum_{k=1}^{n} [5 - (3 + kh)]h$$

and take $U = L + 2h$ where $h = \frac{(5 - 3)}{n} = \frac{2}{n}$.

$$L = h \sum_{k=1}^{n} (2 - kh)$$

$$= h \sum_{k=1}^{n} 2 - h^2 \sum_{k=1}^{n} k$$

$$= 2nh - \frac{h^2 n(n + 1)}{2}$$

$$= 4 - \frac{4(1 + \frac{1}{n})}{2}$$

$$= 2 - \frac{2}{n}$$

It follows that the integral is 2.

2. (a) Find the minimum and the maximum values of $f(x) = 2 + 2x - x^2$ on the interval $[0,1]$, and use them to find two numbers respectively below and above the value of $\int_0^1 f(x)\,dx$.

$f'(x) = 2 - 2x$ is zero when $x = 1$.

$f(0) = 2$ and $f(1) = 3$.

Max: $f(x) = 3$, Min: $f(x) = 2$ for x in $[0,1]$.

Hence $U = 3 \cdot 1 = 3$ and $L = 2 \cdot 1 = 2$.

(b) Check your result by evaluating the integral.

Use the summations of Exercises A5-2, Number 1 and observe that upper and lower sums for x^2 are the negatives of lower and upper sums, respectively, for $-x^2$.

$$\int_0^1 f(x)dx = 2 + 1 - \frac{1}{3} = 3 - \frac{1}{3},$$

a value between 2 and 3.

3. Find upper and lower sums differing by less than .1 for the area under $f(x) = \frac{1}{x}$ on the interval $[1,2]$.

Take a subdivision of $[1,2]$ into n equal subintervals and use the maximum and minimum of $\frac{1}{x}$ as bounds in the subinterval, we have

$$U - L = (1 - \frac{1}{2})\frac{1}{n} = \frac{1}{2n}.$$

It is sufficient to take $\frac{1}{2n} < \frac{1}{10}$ or $n > 5$.
Taking $n = 6$ we obtain

$$U = (1 + \frac{6}{7} + \frac{6}{8} + \frac{6}{9} + \frac{6}{10} + \frac{6}{11}) \cdot \frac{1}{6} = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots + \frac{1}{11},$$

$$L = (\frac{6}{7} + \frac{6}{8} + \frac{6}{9} + \frac{6}{10} + \frac{6}{11} + \frac{6}{12}) \cdot \frac{1}{6} = \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \dots + \frac{1}{12}.$$

(A decimal representation of the answer is not required.)

4. Evaluate each of the following integrals, using upper and lower sum estimates.

(a) $\int_{-1}^{1} x^3 \, dx$

(b) $\int_{-2}^{2} |x| \, dx$

(c) $\int_{-1}^{1} x^2 \, dx$

In each of the following use a subdivision into $2n$ equal parts, and set $h = \frac{(b - a)}{2n}$ where a and b are the lower and upper ends of integration respectively, and separate the sums for positive and negative values of x.

(a) Separate the upper sums into two sums over the intervals $[-1,0]$ and $[0,1]$.

$$U = \sum_{k=1}^{n} (kh)^3 h - \sum_{i=1}^{n} [(i - 1)^3 h^3] h$$

$$= \sum_{k=1}^{n} (kh)^3 h - \sum_{k=0}^{n-1} (kh)^3 h$$

$$= (nh)^3 h$$

$$= h \qquad \text{(from } h = \frac{2}{2n} = \frac{1}{n}\text{).}$$

Then

$$L = U - 2h = -h.$$

The integral is zero.

(b) Separate the upper sums into sums over the intervals $[-2,0]$, $[0,2]$.

$$U = \sum_{k=1}^{n} |kh| h + \sum_{k=1}^{n} |-kh| h$$

$$= 2 \sum_{k=1}^{n} (kh) h$$

$$= 2h^2 \sum_{k=1}^{n} k$$

$$= h^2 n(n + 1) \qquad \text{(where } hn = 2\text{)}$$

$$= 4 + 2h$$

and

$$L = U - 2h - 2h = 4 - 2h.$$

The value of the integral is 4.

(c) By the method of the preceding exercise we find

$$U = 2\sum_{k=1}^{n} (kh)^2\, h = 2h^3 \sum_{k=1}^{n} k^2$$

and since $h = \frac{1}{n}$, from Section 6-2, we have

$$U = \frac{2}{n^3}\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)$$

$$= \frac{2}{3} + \frac{1}{n} + \frac{1}{3n^2}$$

$$L = U - 2h = U - \frac{2}{n}.$$

The integral is $\frac{2}{3}$.

5. Approximate $\int_0^1 \frac{1}{1 + x^2}\, dx$ by Riemann sums.

Given a subdivision $\sigma = \{x_0, x_1, \ldots, x_n\}$ of $[0,1]$ we have

$$R = \sum_{k=1}^{n} \frac{1}{1 + \xi_k^{\,2}}(x_k - x_{k-1})$$

where $x_{k-1} \le \xi_k \le x_k$. If we choose an equal subdivision and take $\xi_k = x_k$ we obtain the following approximations:

$$n = 1 \qquad \frac{1}{1 + 1}\cdot 1 = \frac{1}{2} = 0.50$$

$$n = 2 \qquad \left(\frac{1}{1 + \frac{1}{4}} + \frac{1}{1 + 1}\right)\frac{1}{2} = 0.65$$

$$n = 3 \qquad \left(\frac{1}{1 + \frac{1}{9}} + \frac{1}{1 + \frac{4}{9}} + \frac{1}{1 + 1}\right)\frac{1}{3} \approx 0.70$$

(Exact value is $\frac{\pi}{4} = 0.785 \ldots$).

6. A function f defined on the interval $[a,b]$ is said to be a step-function on $[a,b]$ if for some partition $\sigma = \{x_0, x_1, \ldots, x_n\}$ of the interval, $f(x)$ is constant on each open subinterval (x_{k-1}, x_k), $k = 1, 2, \ldots, n$. Thus $\operatorname{sgn} x$ is a step function on $[-1,1]$, where $\operatorname{sgn} x$ is defined by

$$\operatorname{sgn} x = \begin{cases} -1 , & x < 0 \\ 0 , & x = 0 \\ 1 , & x > 0. \end{cases}$$

Find $\displaystyle\int_a^b \operatorname{sgn} x \, dx.$

If $[a,b]$ does not include the origin there are two cases.

(i) $a > 0$. Since $\operatorname{sgn} x = 1$ on $[a,b]$, the integral has the value $b - a$.

(ii) $b < 0$. Since $\operatorname{sgn} x = -1$ on $[a,b]$ the integral has the value $-(b - a)$.

(iii) If $[a,b]$ includes the origin, $a \leq 0 \leq b$. We take a subdivision and upper and lower sums and obtain the integral I for each case as follows.

$a = 0$, $a = \{0, \epsilon, b\}$,

$$U = b, \quad L = b - \epsilon$$

$$I = b$$

$a < 0 < b$, $\sigma = \{a, -\epsilon, \epsilon, b\}$, where $0 < \epsilon < \min\{-a, b\}$,

$$U = b + a + 2\epsilon$$

$$L = b + a - 2\epsilon$$

$$I = b + a$$

$b = 0$, $\sigma = \{a, -\epsilon, 0\}$

$$U = a + \epsilon, \quad L = \text{[illegible]}$$

$$I = a$$

These results can be summarized by the simple formula

$$\int_a^b \operatorname{sgn} x \, dx = |b| - |a|.$$

7. Evaluate each of the following integrals: The function $[x]$ is defined in Appendix 1.

(a) $\int_{-1}^{3} [\![3x + 4]\!]\,dx$ (c) $\int_{1}^{5} \sqrt{2[\![x]\!]}\,dx$

(b) $\int_{0}^{10} [\![\frac{x}{4}]\!]\,dx$ (d) $\int_{1}^{5} [\![\sqrt{2x}]\!]\,dx$

Each of the given integrands is an increasing step-function and hence is integrable either by the monotone property or by Number 6. In the notation of the solution of Number 6a, the integral of a step-function is

$$\sum_{k=1}^{n} c_k(x_k - x_{k-1})$$

as can be deduced directly from the given upper and lower estimates. Apply this result as follows.

(a) $\int_{-1}^{3} [\![3x + 4]\!]dx = (1 + 2 + 3 + \dots + 12) \cdot \frac{1}{3} = 26$

(b) $\int_{0}^{10} [\![\frac{x}{4}]\!]dx = (0 + 1) \cdot 4 + 2 \cdot 2 = 8$

(c) $\int_{1}^{5} \sqrt{2[\![x]\!]}\,dx = (\sqrt{2} + \sqrt{4} + \sqrt{6} + \sqrt{8}) \cdot 1$

$$= 2 + 3\sqrt{2} + \sqrt{6}$$

(d) $\int_{1}^{5} [\![\sqrt{2x}]\!]dx = 1 \cdot 1 + 2 \cdot \frac{5}{2} + 3 \cdot \frac{1}{2} = \frac{15}{2}$

8. Show that $\int_{a}^{a} f(x)dx = 0$.

The integral can be calculated by subdividing the interval $[a,a]$, and calculating the appropriate Riemann sums. But all subintervals of $[a,a]$ are of length zero, and any element of the Riemann sum,

$$f(x_k)[a - a] = 0.$$

Alternately, the Fundamental Theorem of Calculus states that

$$\int_{a}^{a} f(x)dx = F(a) - F(a) = 0$$

where $F'(x) = f(x)$.

Solutions Exercises A5-5

1. Exhibit the details of the proof of Part (i) when $\alpha < 0$.

If $m_k \leq f(x) \leq M_k$ on $[x_{k-1}, x_k]$ and $\alpha < 0$, then multiply by α to obtain

$$\alpha M_k \leq \alpha f(x) \leq \alpha m_k.$$

From $\alpha \sum_{k=1}^{n} M_k(x_k - x_{k-1}) \leq \alpha \sum_{k=1}^{n} m_k(x_k - x_{k-1})$ obtain the lower sum αL and upper sum αU for f over $[a,b]$. If $U - L < \epsilon$, then

$$0 < \alpha L - \alpha U < |\alpha|\epsilon.$$

Observe that αf is then integrable by Theorem A5-4a. If I is the integral of f and J that of αf over $[a,b]$, then

$$0 \leq U - I < \epsilon, \quad 0 \leq J - \alpha U < |\alpha|\epsilon,$$

from which it follows that

$$\begin{aligned} |J - \alpha I| &= |(J - \alpha U) + \alpha(U - I)| \\ &\leq |J - \alpha U| + |\alpha| \cdot |U - I| \\ &< 2|\alpha|\epsilon. \end{aligned}$$

This result holds for all positive ϵ, hence $J = \alpha I$.

2. (a) If the graph of f is symmetric with respect to the origin, then f is odd. Prove that if f is odd and integrable on $[-a,a]$, then

$$\int_{-a}^{a} f(x)dx = 0.$$

(b) If the graph of f is symmetric with respect to the y-axis, then f is even. Prove for an even function f which is integrable on $[-a,a]$ that

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

Interpret this result geometrically.

Take a subdivision of the interval into 2n equal parts and set $h = \frac{a}{n}$. Take the Riemann sum

$$R = \sum_{k=1}^{n} f(-\xi_k)h + \sum_{k=1}^{n} f(\xi_k)h.$$

where $(k - 1)h \le \xi_k \le kh$, $(k = 1, \ldots, n)$.

(a) If f is odd, $f(-x) = -f(x)$, and

$$R = \sum_{k=1}^{n} [-f(\xi_k)h] + \sum_{k=1}^{n} f(\xi_k)h$$

$$= 0.$$

Since the limit of the Riemann sums is the same independently of the method of subdivision and the choice of ξ_k it follows that the integral is 0.

(b) If f is even, $f(-x) = f(x)$, and $R = 2\sum_{k=1}^{n} f(\xi_k)h$ where the sum is the Riemann sum for f over the half interval $[0,a]$, and the result follows on taking the limit. Geometrical interpretation: The area of a standard region is equal to that of its mirror image. Parts (a) and (b) can also be done by comparing upper and lower sums on the half-intervals.

3. Prove Theorem A5-5c as a consequence of Lemmas A5-5a and A5-5b. Conversely, derive the Lemmas as corollaries of Theorem A5-5c.

Proof of Theorem A5-5c.
Let f and g be integrable over $[a,b]$. From Lemmas A5-5a and A5-5b applied in succession

$$\alpha\int_a^b f(x)dx + \beta\int_a^b g(x)dx = \int_a^b \alpha f(x)dx + \int_a^b \beta\, g(x)dx$$

$$= \int_a^b [\alpha\, f(x) + \beta\, g(x)]dx.$$

Proof of Lemma A5-5a. Take $\beta = 0$ in Theorem A5-5a.
Proof of Lemma A5-5b. Take $\alpha = \beta = 1$ in Theorem A5-5a.

4. Prove: If f and g are integrable where $g : x \to |f(x)|$ on $[a,b]$. then

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$$

For all x in $[a,b]$

$$-|f(x)| \leq f(x) \leq |f(x)|,$$

whence by Theorem A5-5a

$$\int_a^b -|f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx$$

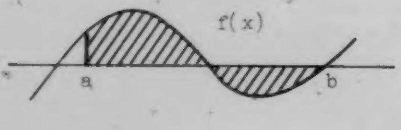
or

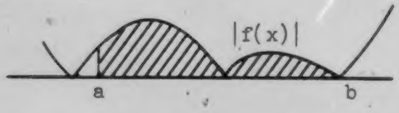
$$-\int_a^b |f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx$$

or

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$$

We observe that $\int_a^b |f(x)|dx$ represents the sum of the areas of the regions bounded by the graph of f, the x-axis, and the vertical lines, $x = a$, $x = b$.





5. Compute the values of the given integrals using Theorem A5-5c.

(a) $\int_2^3 (3x^2 - 5x + 1)dx$

$$\int_2^3 (3x^2 - 5x + 1)dx = 3\int_2^3 x^2\,dx - 5\int_2^3 x\,dx + \int_2^3 dx.$$

From Examples A5-5a, b, c, and

$$\int_a^b f(x)dx = \int_0^b f(x)dx - \int_0^a f(x)dx$$

it follows that the integral is

$$3\left(\frac{3^3}{3} - \frac{2^3}{3}\right) - 5\left(\frac{3^2}{2} - \frac{2^2}{2}\right) + (3 - 2) = 7\frac{1}{2} .$$

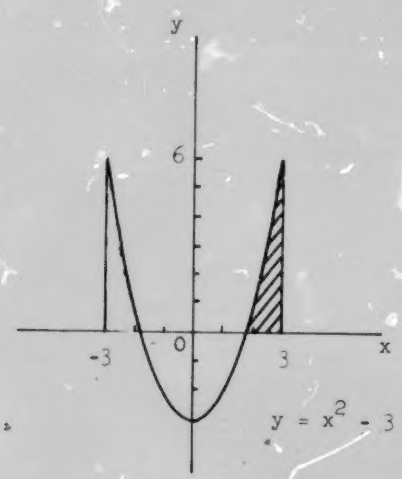
(b) $\int_0^2 (x - 1)(x + 2)dx$

$$\begin{aligned}\int_0^2 (x - 1)(x + 2)dx &= \int_0^2 (x^2 + x - 2)dx \\ &= \frac{2^3}{3} + \frac{2^2}{2} - 2(2) \\ &= \frac{2}{3} .\end{aligned}$$

(c) $\int_{-2}^3 (x + 2)(x - 3)dx$

$$\begin{aligned}\int_{-2}^3 (x + 2)(x - 3)dx &= \int_{-2}^3 (x^2 - x - 6)dx \\ &= \left(\frac{3^3}{3} - \frac{(-2)^3}{3}\right) - \left(\frac{3^2}{2} - \frac{(-2)^2}{2}\right) - 6\big(3 - (-2)\big) \\ &= -\frac{125}{6} .\end{aligned}$$

6. (a) Find the area of the region below the parabola $y = x^2 - 3$ above the x-axis and between the lines $x = -3$, $x = 3$.

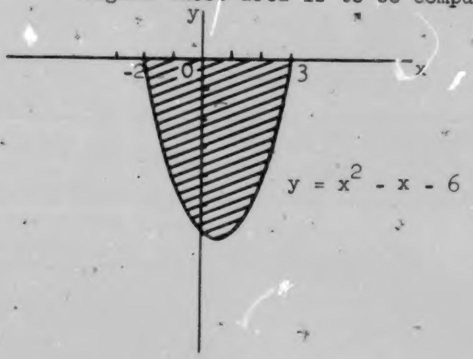


$$\text{Area} = 2\int_{\sqrt{3}}^{3}(x^2 - 3)dx$$

$$= 2\left[\left(\frac{3^3}{3} - \frac{(\sqrt{3})^3}{3}\right) - 3(3 - \sqrt{3})\right]$$

$$= 4\sqrt{3}..$$

(b) Find the area of the region between the graph of $f : x \to x^2 - x - 6$, the x-axis, and the lines $x = -2$, $x = 3$. First draw a rough sketch of f and indicate (by shading) the region whose area is to be computed.



$$\text{Area} = -\int_{-2}^{3}(x^2 - x - 6)dx$$

$$= \frac{125}{6}.$$

(See No. 7c.)

7. Find all values of a for which

$$\int_0^a (x + x^2)dx = 0.$$

The number a must satisfy $\frac{a^3}{3} + \frac{a^2}{2} = 0$. This equation has two solutions

$$a = -\frac{3}{2} \text{ and } a = 0.$$

8. Compute $\int_0^3 f(x)dx$ where

$$f(x) = \begin{cases} 2 - x^2, & 0 \leq x \leq 1 \\ 5 - 4x, & 1 \leq x \leq 3 \end{cases}$$

$$\int_0^3 f(x)dx = \int_0^1 (2 - x^2)dx + \int_1^3 (5 - 4x)dx$$

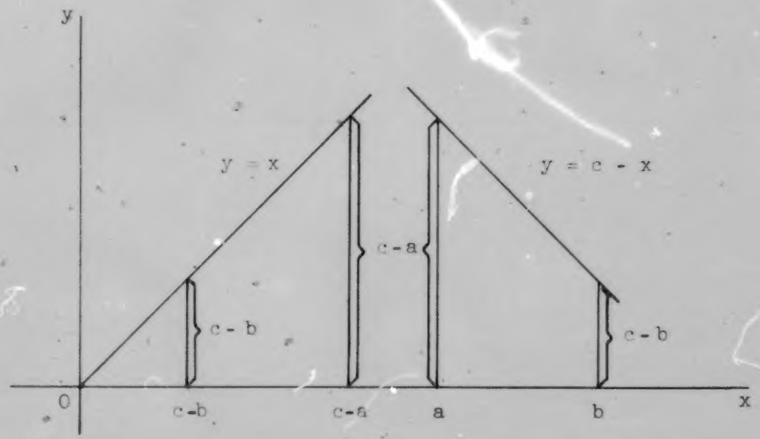
$$= (2 - \frac{1}{3}) + [10 - 4(\frac{3^2}{2} - \frac{1}{2})]$$

$$= \frac{-13}{3} .$$

9. Verify that the following property holds for $f : x \rightarrow x$

$$\int_a^b f(c - x)dx = \int_{c-b}^{c-a} f(x)dx.$$

Explain the property geometrically in terms of areas. Do you think that the property holds for other functions that are integrable? Justify your answer.



The integrals represent areas of mirror-image regions. The property is general for a function f integrable over $[c - b, c - a]$. For $f(x) = x$, particularly,

$$\int_a^b (c - x)dx = c(b - a) - \frac{1}{2}(b^2 - a^2),$$

$$\begin{aligned}\int_{c-b}^{c-a} x\,dx &= \frac{1}{2}\left[(c - a)^2 - (c - b)^2\right]\\ &= \frac{1}{2}\left[2bc - 2ac + a^2 - b^2\right]\\ &= c(b - a) - \frac{1}{2}(b^2 - a^2).\end{aligned}$$

10. If a function f is periodic with period λ and integrable for all x, show that

$$\int_a^{a+n\lambda} f(x)dx = n\int_a^{a+\lambda} f(x)dx$$

(n, integer). Interpret geometrically.

$$\int_a^{a+n\lambda} f(x)dx = \sum_{k=1}^{n} \int_{a+(k-1)\lambda}^{a+k\lambda} f(x)dx.$$

Now consider the subdivision of the interval $[a + (k - 1)\lambda , a + k\lambda]$ into m equal parts by means of the partition $\{u_0, u_1, \ldots, u_m\}$ where $u_i = a + (k - 1)\lambda + \frac{i\lambda}{m}$ and form the Riemann sum

$$\begin{aligned}R_k &= \sum_{i=1}^{m} f(u_i)\frac{\lambda}{m}\\ &= \sum_{i=1}^{n} f\left(a + (k - 1)\lambda + \frac{i\lambda}{m}\right)\frac{\lambda}{m}\\ &= \sum_{i=1}^{n} f\left(a + \frac{i\lambda}{m}\right)\frac{\lambda}{m} .\end{aligned}$$

Since the Riemann sums R_k, $(k = 1 , \ldots , n)$ for each of the integrals are the same, it follows that the integrals over the intervals $[a + (k - 1)\lambda , a + k\lambda]$ are equal and the result follows.

Geometrically, the standard regions for the intervals $[a + (k - 1)\lambda , a + k\lambda]$ are congruent.

11. Evaluate (without using the Fundamental Theorem of Calculus)

$$\int_0^{100\pi} (1 + \sin 2x)dx$$

Note: This exercise uses Exercises A5-2, No. 1(d) which requires Section A3-2(ii).

Since the integrand is periodic with period π, it follows from Number 10 that

$$\int_0^{100\pi} (1 + \sin 2x)dx = 100\int_0^{\pi} (1 + \sin 2x)dx.$$

From Exercises A5-2, Number 1(d)

$$\int_0^{\pi} \sin 2x \, dx = \frac{1 - \cos 2\pi}{2} = 0.$$

Answer: 100π.

12. Prove that if f is integrable on $[a,b]$ and if $f(x) \geq 0$ for all x in $[a,b]$, then

$$\int_a^b f(x)dx \geq 0.$$

For every partition of $[a,b]$, $L = 0$ is a lower sum. Since the integral is an upper bound for all lower sums, the result follows. The result is also an immediate consequence of Theorem A5-5a.

13. Prove that if f and g are integrable over $[a,b]$, then

$$\left|\int_a^b (g(x) - f(x))dx\right| \leq \int_a^b |g(x)|dx + \int_a^b |f(x)|dx.$$

From Number 4

$$\left|\int_a^b [g(x) - f(x)]dx\right| \leq \int_a^b |g(x) - f(x)|dx.$$

But $|g(x) - f(x)| \leq |g(x)| + |f(x)|$. It follows from Theorem A5-5a and A5-5b that

$$\int_a^b |g(x) - f(x)|dx \leq \int_a^b |g(x)|dx + \int_a^b |f(x)|dx.$$

14. Let f and g be integrable and suppose that $f(x) \leq g(x)$ on $[a,b]$.
(a) If the inequality $f(x) + \epsilon \leq g(x)$,, for some $\epsilon > 0$, holds on any subinterval of $[a,b]$, prove the strong inequality

$$\int_a^b f(x)dx < \int_a^b g(x)dx.$$

Let $[u,v]$ be a subinterval of $[a,b]$ in which $f(x) + \epsilon < g(x)$. We have $g(x) - f(x) > \epsilon$ and by Theorem A5-5a

$$\int_u^v [g(x) - f(x)]dx \geq \int_u^v \epsilon \, dx = \epsilon \, (v - u).$$

Since $g(x) - f(x) \geq 0$ on the rest of the interval we have

$$\int_a^b [g(x) - f(x)]dx = \int_a^u [g(x) - f(x)]dx + \int_u^v [g(x) - f(x)]dx + \int_v^b [g(x) - f(x)]dx$$

$$\geq 0 + \epsilon \, (v - u) + 0$$

whence

$$\int_a^b g(x)dx - \int_a^b f(x)dx \geq \epsilon \, (v - u) > 0$$

from which the conclusion follows.

(b) If f and g are continuous at $x = u$ in $[a,b]$ and $f(u) < g(u)$ prove that strong inequality holds as above.

From the conditions of the problem,

$$g(u) - f(u) = \epsilon > 0.$$

Also, continuity of f and g implies that there is some neighborhood of u in which

$$|f(x) - f(u)| < \frac{\epsilon}{4}$$

and $$|g(x) - g(u)| < \frac{\epsilon}{4}.$$

Combining, we obtain

$$g(x) - f(x) > \frac{\epsilon}{2}$$

in some neighborhood of u. The result follows from part (a).

15. If functions f and g are integrable, and $f(x) \leq h(x) \leq g(x)$ on $[a,b]$, does it follow that

$$\int_a^b f(x)dx \leq \int_a^b h(x)dx \leq \int_a^b g(x)dx?$$

Illustrate by an example.

No! The function h may not be integrable. For example, take $f(x) = -1$, $g(x) = 1$,

$$h(x) = \begin{cases} 1, & x \text{ rational} \\ -1, & x \text{ irrational.} \end{cases}$$

On every interval $\max\{h(x)\} = 1$, $\min\{h(x)\} = -1$, hence for every upper sum, $U \geq (b - a)$, and for every lower sum, $L \leq -(b - a)$. Thus h is not integrable by Theorem A5-4a.

16. (a) Prove the Mean Value Theorem of integral calculus: If f is continuous and integrable on $[a,b]$, then there exists a value u in the open interval (a,b) such that

$$\int_a^b f(x)dx = f(u)(b - a).$$

By the Extreme Value Theorem $f(x)$ takes on a maximum value M and a minimum value m in $[a,b]$. Since

$$m \leq f(x) \leq M$$

on $[a,b]$ we have from Theorem A5-5a

$$\int_a^b m\,dx \leq \int_a^b f(x)dx \leq \int_a^b M\,dx$$

whence

$$m \leq \frac{1}{b - a}\int_a^b f(x)dx \leq M.$$

Since $f(x)$ takes on every value between m and M (Intermediate Value Theorem, there is a value u in $[a,b]$ for which

$$f(u) = \frac{1}{b - a}\int_a^b f(x)dx.$$

(b) Show that the value $f(u)$ in (a) satisfies

$$f(u) = \lim_{h \sim 0} \frac{f_0 + f_1 + \dots + f_n}{n + 1}$$

where $h = \frac{b - a)}{n}$ and $f_k = f(a + kh)$ for $k = 0, 1, 2, \dots,$ n. Thus $f(u)$ can be interpreted as an extension of the idea of mean or arithmetic average to the values of a function on an interval.

The expression for the integral as a limit of Riemann sums is

$$\int_a^b f(x)dx = \lim_{h \sim 0} \sum_{k=1}^{n} f_k h = \lim_{h \sim 0} \sum_{k=0}^{n} f_k h$$

$$= \lim_{h \sim 0} (n + 1)h \left\{ \frac{\sum_{k=0}^{n} f_k}{n + 1} \right\}$$

$$= \lim_{h \sim 0} (b - a + h) \left\{ \frac{\sum_{k=0}^{n} f_k}{n + 1} \right\}$$

$$= \lim_{h \sim 0} (b - a + h) \lim_{h \sim 0} \left\{ \frac{\sum_{k=0}^{n} f_k}{n + 1} \right\}$$

from which the result follows (provided

$$\lim_{h \sim 0} \left\{ \frac{\sum_{k=0}^{n} f_k}{n + 1} \right\}$$

exists.) Existence is a consequence of the fact that if $\lim_{h \sim 0} pq$ exists and $\lim_{h \sim 0} p$ exists but $\lim_{h \sim 0} p \neq 0$ then $\lim_{h \sim 0} q$ exists and

$$\lim_{h \sim 0} q = \frac{\lim_{h \sim 0} pq}{\lim_{h \sim 0} p} .$$

(The point need not be brought up unless the issue of existence is raised in class.)

17. If $\frac{a_0}{n + 1} + \frac{a_1}{n} + \dots + \frac{a_{n-1}}{2} + \frac{a_n}{1} = 0$, show that

$$a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$$

has at least one real root in $(0,1)$.

Set $f(x) = \sum_{k=0}^{n} a_k x^{n-k}$. By the Mean Value Theorem (No. 16a) there is a point u in $(0,1)$ for which,

$$\begin{aligned} f(u) &= \int_0^1 f(x)dx \\ &= \frac{a_0x^{n+1}}{n + 1} + \frac{a_1x^n}{n} + \dots + \frac{a_nx}{1}\Bigg|_0^1 \\ &= \frac{a_0}{n + 1} + \frac{a_1}{n} + \dots + \frac{a_n}{1} \\ &= 0. \end{aligned}$$

18. (a) Prove that if $f(x)$ is integrable on $[a,b]$, then $|f(x)|$ is integrable on $[a,b]$.

For each positive ϵ there exist a partition $\sigma = \{x_1, x_2, x_3, \dots, x_n\}$, an upper sum U, and a lower sum L over σ for which

$$U - L = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1}) < \epsilon,$$

where $m_k \leq f(x) \leq M_k$ on $[x_{k-1},x_k]$. We shall describe upper and lower bounds M_k^* and m_k^* for $|f(x)|$ on each interval $[x_{k-1},x_k]$. If $m_k \geq 0$, then $|f(x)| = f(x)$ and we take $M_k^* = M_k$ and $m_k^* = m_k$ on $[x_{k-1},x_k]$. If $M_k \leq 0$, then $|f(x)| = -f(x)$ and we take $M_k^* = -m_k$ and $m_k^* = -M_k$; whence

$$M_k^* - m_k^* = -m_k - (-M_k) = M_k - m_k.$$

If $M_k \geq 0$ and $m_k < 0$ we consider two cases:

(i) $|m_k| \le M_k$. Taking $M_k^* = M_k$ and $m_k^* = 0$, we have

$$M_k^* - m_k^* = M_k < M_k + |m_k| \le M_k - m_k.$$

(ii) $|m_k| > M_k$. Taking $M_k^* = |m_k|$ and $m_k^* = 0$, we have

$$M_k^* - m_k^* = |m_k| \le |m_k| + M_k \le M_k - m_k.$$

In every case

$$U^* - L^* \le U - L < \epsilon$$

where U^* and L^* are the upper and lower sums for $|f(x)|$ constructed by use of the bounds M_k^* and m_k^*.

Alternate Solution

Define f^+ and f^- by

$$f^+(x) = \begin{cases} f(x), & \text{if } f(x) \ge 0, \\ 0, & \text{if } f(x) \le 0, \end{cases} \qquad f^-(x) = \begin{cases} -f(x), & \text{if } f(x) \le 0, \\ 0, & \text{if } f(x) \ge 0. \end{cases}$$

Since $|f| = f^+ + f^-$, it is necessary only to show that f^+ and f^- are integrable in order to prove that $|f|$ is integrable. Now, for each positive ϵ, there exist a partition σ, an upper sum U, and a lower sum L for f over σ such that, in the notation of the text,

$$U - L = \sum (M_k - m_k)(x_k - x_{k-1}) < \epsilon.$$

On each subinterval $[x_{k-1}, x_k]$ choose upper and lower bounds M^+ and m^+ for f^+ as follows. If $f(x) \ge 0$ for at least one point x in $[x_{k-1}, x_k]$ take $M_k^+ = M_k$ and $m_k^+ = m_k$; if $f(x) < 0$ everywhere on the interval, take $M_k^+ = m_k^+ = 0$. In either case

$$M_k^+ - m_k^+ \le M_k - m_k.$$

Similarly, for f^- the corresponding upper and lower bounds satisfy

$$M_k^- - m_k^- \le M_k - m_k.$$

It follows for the corresponding upper and lower sums that

$$U^+ - L^+ < \epsilon, \quad U^- - L^- < \epsilon,$$

Consequently, f^+ and f^- are integrable and so also is $|f|$.

19. Suppose

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

Show that if U and L are upper and lower sums for a partition of $[0,1]$ then $U \geq 1$ and $L \leq -1$. Is f integrable on $[0,1]$?

See Number 15.

Note that $|f(x)| = 1$ on $[0,1]$ and that

$$\int_0^1 |f(x)|dx = 1.$$

Thus, the fact that $|f(x)|$ is integrable on $[a,b]$ does not imply that $f(x)$ is integrable on $[a,b]$.

20. If f and g are integrable on $[a,b]$ then both $\max\{f,g\}$ and $\min\{f,g\}$ are also integrable on $[a,b]$.

Use Number 19 and the results of Exercises A1-3, Number 7:

$$\max\{f,g\} = \frac{1}{2}(f + g + |f - g|),$$

$$\min\{f,g\} = \frac{1}{2}(f + g - |f - g|);$$

the result follows from Lemma A5-5b.

Alternate Solution

There exists a partition σ and upper and lower sums U_1 and L_1 for f, U_2 and L_2 for g, over σ such that both

$$U_1 - L_1 < \epsilon \quad \text{and} \quad U_2 - L_2 < \epsilon.$$

(For example, if σ_1 is such a partition for f, σ_2 for g, take $\sigma = \sigma_1 \cup \sigma_2$.) Let M_k', m_k' denote the bounds for f in the expressions for U_1 and L_1, M_k'' and m_k'' the corresponding bounds for g. Set $\Phi = \max\{f,g\}$. On each interval $[x_{k-1},x_k]$ choose as the upper bound for Φ, $M_k = \max\{M_k',M_k''\}$; as the lower bound, choose

$$m_k = \begin{cases} m_k' , & \text{if } M_k = M_k', \\ m_k'' , & \text{otherwise.} \end{cases}$$

With this choice,

$$\begin{aligned} M_k - m_k &\leq \max\{M_k' - m_k', M_k'' - m_k''\} \\ &\leq (M_k' - m_k') + (M_k'' - m_k''). \end{aligned}$$

Form the appropriate upper and lower sums U and L for Φ to obtain

$$U - L \leq (U_1 - L_1) + (U_2 - L_2) < 2\epsilon$$

from which the conclusion follows.

To obtain the result for $\min\{f,g\}$ observe that $\min\{f,g\} = -\max\{-f,-g\}$.

Finally, since $|f| = \max\{f,-f\}$, this proof also serves to demonstrate Number 21.

21. Let f and g be bounded and integrable on $[a,b]$.
(a) Prove $f \cdot g$ is integrable on $[a,b]$.

First observe that it is sufficient to prove the result when f and g are positive. This is true from the boundedness of f and g, because there exist constants c_1 and c_2 such that $\phi = f + c_1$ and $\psi = g + c_2$ are positive. If the result is true for positive functions then $\phi \cdot \psi = f \cdot g + c_1 g + c_2 f + c_1 c_2$ is integrable, hence $f \cdot g$ is integrable.

Now let σ be a partition, M_k' and m_k' upper and lower bounds for f, M_k'' and m_k'' for g, on the subinterval $[x_{k-1},x_k]$ of the partition, and let U', L' and U'', L'' be the corresponding upper and lower sums. For a sufficiently fine partition and appropriate choice of bounds,

$$U' - L' < \epsilon \quad \text{and} \quad U'' - L'' < \epsilon.$$

Now, choose the upper and lower bounds $M_k = M_k' M_k''$, $m_k = m_k' m_k''$ for $f \cdot g$. Then

$$\begin{aligned} M_k - m_k &= M_k'(M_k'' - m_k'') + m_k''(M_k' - m_k') \\ &\leq A(M_k'' - m_k'') + B(M_k' - m_k') \end{aligned}$$

where A and B are overall upper bounds for f and g, respectively. Form the appropriate upper and lower sums for $f \cdot g$ over σ to obtain

$$\begin{aligned} U - L &\leq A(U' - L') + B(U'' - L'') \\ &< \epsilon(A + B), \end{aligned}$$

from which it follows that $f \cdot g$ is integrable.

(b) If g is bounded away from zero, then $\frac{f}{g}$ is integrable over $[a,b]$.

For the proof, it is sufficient to prove $\frac{1}{g}$ is integrable and then to apply Part (a). Now suppose $|g(x)| \geq c > 0$. Let U and L be upper and lower sums for g over σ such that $U - L < \epsilon$, and let M_k, m_k denote upper and lower bounds for g on the subinterval $I_k = [x_{k-1}, x_k]$. Choose upper and lower bounds M_k^* and m_k^* for $\frac{1}{g}$ as follows. If $m_k \geq c$ or $M_k \leq -c$, then take $M_k^* = \frac{1}{m_k}$ and $m_k^* = \frac{1}{M_k}$. In this case, $M_k^* - m_k^* = \frac{M_k - m_k}{m_k M_k} \leq \frac{M_k - m_k}{c^2}$. If $m_k \leq -c$ and $M_k \geq c$, then take $m_k^* = -\frac{1}{c}$ and $M_k^* = \frac{1}{c}$. In this case use $1 \leq \frac{M_k}{c}$ and $1 \leq -\frac{m_k}{c}$ to obtain $M_k^* - m_k^* = \frac{1}{c} + \frac{1}{c} \leq \frac{M_k}{c^2} - \frac{m_k}{c^2} \leq \frac{M_k - m_k}{c^2}$. For the corresponding upper and lower sums U^* and L^*, then

$$U^* - L^* \leq \frac{U - L}{c^2} < \frac{\epsilon}{c^2}.$$

22. If f and g are bounded and integrable, then $\int_a^b \left(\alpha f(x) + \beta g(x)\right)^2 dx$ exists and is greater than or equal to 0 for all constant α and β. Show from this that

$$\int_a^b f(x)^2 dx \int_a^b g(x)^2 dx \geq \left\{ \int_a^b f(x)g(x)\,dx \right\}^2$$

with equality if and only if (for f and g continuous) $f = cg$ on $[a,b]$ for some constant c.

Consider the inequality

$$\int_a^b \left(f(x) + tg(x)\right)^2 dx = \int_a^b f(x)^2 dx + 2t \int_a^b f(x)g(x)\,dx + t^2 \int_a^b g(x)^2 dx$$
$$\geq 0.$$

An inequality of the form $At^2 + 2Bt + C > 0$ holds for all t if and only if $B^2 - AC \leq 0$, but that is precisely Buniakowsky's inequality in this case.

If $f = cg$ then equality obviously holds. If equality holds $B^2 - AC = 0$, then for some choice of t, say $t = -c$

$$\int_a^b [f(x) - cg(x)]^2 dx = 0.$$

Now $f - cg$ must be identically zero, for if there were any point where $f(x_0) - cg(x_0) \neq 0$, then $[f(x_0) - cg(x_0)]^2 > 0$ and from the continuity of f and g there would be an interval containing x_0 where the integrand has a positive lower bound. In that case the integral would have to be positive, not zero as required. Consequently, $f = cg$ is the only possibility. (Note that the proof requires the continuity of f and g **at only one point.**)

23. If f is integrable and its graph is convex on the interval $[0,a]$, show that

$$\int_0^a f(x)dx \geq a\, f(\tfrac{a}{2}).$$

Interpret geometrically.

The graph of f lies above its tangent at $\frac{a}{2}$:

$$f(x) \geq f(\tfrac{a}{2}) + f'(\tfrac{a}{2})(x - \tfrac{a}{2})$$

(if f is not differentiable there still exists a "line of support" at $\frac{a}{2}$ and $f'(\frac{a}{2})$ would be replaced by the slope m of a line of support). Then

$$\int_0^a f(x)dx \geq \int_0^a [f(\tfrac{a}{2}) + f'(\tfrac{a}{2})(x - \tfrac{a}{2})]dx$$

$$\geq \left[f(\tfrac{a}{2})x + f'(\tfrac{a}{2})\left(\frac{x^2}{2} - \frac{ax}{2}\right)\right]\Bigg|_0^a$$

$$\geq a\, f(\tfrac{a}{2}).$$

For f positive, the geometrical intepretation is that the area of the standard region under the graph of f is greater than the area under the tangent taken at the midpoint of the base interval.

24. Show that

$$\sqrt{(a^2 + \tfrac{1}{3})(b^2 + \tfrac{1}{3})} \geq \int_0^1 \sqrt{(x^2 + a^2)(x^2 + b^2)}\, dx$$

Set $f(x) = \sqrt{x^2 + a^2}$, $g(x) = \sqrt{x^2 + b^2}$ in the Buniakowsky inequality (No. 22).

25. Show that

(a) $\frac{1}{2} + \frac{3\sqrt{2}}{8} < \int_0^1 \sqrt{1 + x^3}\, dx < \frac{\sqrt{5}}{2}$.

Take $f(x) = 1$, $g(x) = \sqrt{1 + x^3}$ in the Buniakowsky inequality to obtain the upper estimate. For the lower estimate, use

$$\int_0^1 \sqrt{1 + x^3}\, dx = \int_0^{1/2} \sqrt{1 + x^3}\, dx + \int_{1/2}^1 \sqrt{1 + x^3}\, dx$$

$$> \int_0^{1/2} 1\, dx + \int_{1/2}^1 \sqrt{1 + \tfrac{1}{8}}\, dx.$$

(b) Show that $\frac{1}{2} + \frac{\sqrt{2}}{3} > \int_0^1 \frac{dx}{\sqrt{1 + x^3}} > \frac{2\sqrt{5}}{5}$.

For the upper estimate, use

$$\int_0^1 \frac{dx}{\sqrt{1 + x^3}} = \int_0^{1/2} \frac{dx}{\sqrt{1 + x^3}} + \int_{1/2}^1 \frac{dx}{\sqrt{1 + x^3}}$$

$$< \int_0^1 1\, dx + \int_{1/2}^1 \frac{dx}{\sqrt{1 + \frac{1}{8}}}.$$

For the lower estimate use the Buniakowsky inequality:

$$\int_0^1 \sqrt{1 + x^3}\, dx \int_0^1 \frac{dx}{\sqrt{1 + x^3}} > 1,$$

hence, from Part (a),

$$\frac{\sqrt{5}}{2} \int_0^1 \frac{dx}{\sqrt{1 + x^3}} > 1.$$

26. Find a continuously differentiable function F in [0,1] such that
(a) $F(0) = 0$, $F(1) = a$,
(b) $\int_0^1 F(x)^2 dx = \frac{a^2}{3}$,
(c) $\int_0^1 F'(x)^2 dx$ is a minimum.

Take $f(x) = 1$ and $g(x) = F'(x)$ in the Buniakowsky inequality (No. 22):

$$\int_0^1 1\,dx \int_0^1 F'(x)^2 dx \geq \left\{\int_0^1 F'(x)dx\right\}^2$$
$$\geq (F(1) - F(0))^2$$
$$\geq a^2.$$

Equality holds when $g = cf$ for constant c. Thus $F'(x) = c$, hence $F(x) = cx + d$. From condition (a), $F(x) = ax$; condition (b) is automatically satisfied and is therefore redundent. Condition (c) is satisfied since equality is achieved in the inequality above.

Appendix 6

TC A6-1. Absolute Value and Inequality

In Section A1-1 (footnote), we define $|a|$ as $\sqrt{a^2}$. This definition has the virtue of emphasizing the positivity of the square root. It also helps to prevent the error of writing $\sqrt{a^2} = a$ in case $a < 0$. This error leads to the amusing "proof":

$$1 = \sqrt{1} = \sqrt{(-1)^2} = -1$$

thus

$$1 = -1.$$

We note that the form $\sqrt{a^2}$ lends itself, more conveniently, to mathematical manipulation.

Solutions Exercises A6-1

1. Find the absolute value of the following numbers.

(a) -1.75	1.75
(b) $-\frac{\pi}{4}$	$\frac{\pi}{4}$
(c) $\sin(-\frac{\pi}{4})$	$\frac{\sqrt{2}}{2}$
(d) $\cos(-\frac{\pi}{2})$	0

2. (a) For what real numbers x does $\sqrt{x^2} = -x$?

   $x \leq 0$

   (b) For what real numbers x does $|1 - x| = x - 1$?

   $x \geq 1$

3. Solve the equations:

(a)	$\|3 - x\| = 1$	$x = 2$ or $x = 4$.
(b)	$\|4x + 3\| = 1$	$x = -\frac{1}{2}$ or $x = -1$.
(c)	$\|x + 2\| = x$	Either $x + 2 \geq 0$ or $x + 2 < 0$, then $x + 2 = x$ or $-(x + 2) = x$. Thus there are no solutions.
(d)	$\|x + 1\| = \|x - 3\|$	The only solution is $x = 1$.
(e)	$\|2x + 5\| + \|5x + 2\| = 0$	There are no solutions.
(f)	$\|2x + 3\| = \|5 - x\|$	$x = -8$ or $x = \frac{2}{3}$.
(g)	$2\|3x + 4\| + \|x - 2\| = 1 + \|3 + x\|$	There are no solutions.

4. For what values of x is each of the following true? (Express your answer in terms of inequalities satisfied by x.)

(a)	$\|x\| \leq 0$	$x = 0$
(b)	$\|x\| \neq x$	$x < 0$
(c)	$\|x\| < 3$	$-3 < x < 3$
(d)	$\|x - 6\| \leq 1$	$5 \leq x \leq 7$
(e)	$\|x - 3\| > 2$	$x < 1$ or $x > 5$
(f)	$\|2x - 3\| < 1$	$1 < x < 2$
(g)	$\|x - a\| < a$	$0 < x < 2a$
(h)	$\|x^2 - 3\| < 1$	$\sqrt{2} < x < 2$ or $-2 < x < -\sqrt{2}$
(i)	$\|(x - 2)(x - 3)\| > 2$	$x < 1$ or $x > 4$
(j)	$\|x - 1\| > \| x - 3\|$	$x > 2$
(k)	$\|x - 5\| + 1 = \|x + 5\|$	$x = \frac{1}{2}$

(ℓ) $\|x - 1\| + \|x - 2\| = 1$	$1 \leq x \leq 2$
(m) $\|x^2 - a^2\| > 0$	$x \neq \pm a$
(n) $\|x - a\| < \delta$	$a - \delta < x < a + \delta$
(o) $0 < \|x - a\| < \delta$	$a - \delta < x < a$ or $a < x < a + \delta$
(p) $\|x - 1\| < 2$ and $\|x + 1\| < \frac{3}{2}$	$-1 < x < \frac{1}{2}$
(q) $\|x - 1\| < 2$ and $\|2x - 1\| < \frac{3}{2}$	$-\frac{1}{4} < x < \frac{5}{4}$
(r) $\|x + y\| = \|x\| + \|y\|$, for all y	$x = 0$
(s) $\|\sin x\| = 0$	$x = n\pi$, n, an integer
(t) $\|\sin x\| > \frac{\sqrt{2}}{2}$	$\frac{\pi}{4} + \pi n < x < \frac{3\pi}{4} + \pi n$
(u) $\|1 - \frac{1}{x}\| < 1$	$x > \frac{1}{2}$
(v) $\sqrt{\|x\|} > \frac{1}{2}$	$x < -\frac{1}{4}$ or $x > \frac{1}{4}$

5. Sketch the graphs of the following equations:

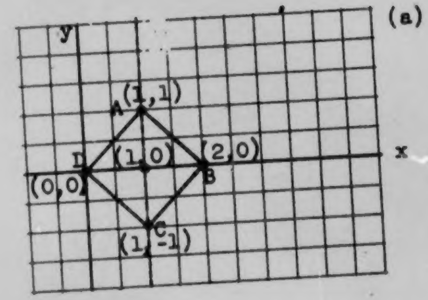
(a) $|x - 1| + |y| = 1$

For $x \geq 1$, $y > 0$, then
$x - 1 + y = 1$ or $x + y = 2$,
line AB.

For $x \geq 1$, $y < 0$, then
$x - 1 - y = 1$ or $x - y = 2$,
line BC.

For $x < 1$, $y < 0$, then
$-x + 1 - y = 1$ or
$x + y = 0$, line CD.

For $x < 1$, $y \geq 0$, then
$-x + 1 + y = 1$, or
$y = x$, line DA.



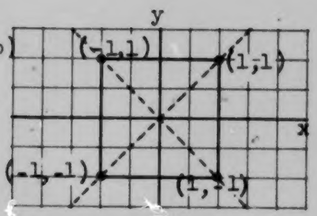
(b) $|x + y| + |x - y| = 2$

Resolves into 4 parts:

$x = \pm 1$ and $y = \pm 1$

where $|x| \leq 1$ and $|y| \leq 1$

(b)



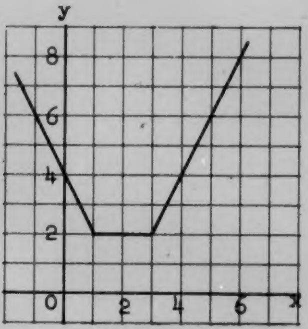
(c) $y = |x - 1| + |x - 3|$

For $x < 1$, then $y = -2x + 4$.

For $1 \leq x \leq w$, then $y = 2$.

For $x > 3$, then $y = 2x - 4$.

(c)



(d) $y = |x - 1| + |x - 3| + 2|x - 4|$

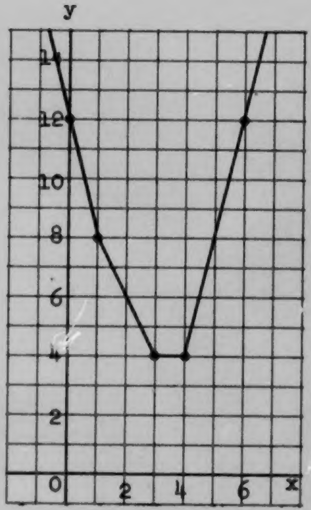
For $x < 1$, then $y = -4x + 12$.

For $1 \leq x < 3$, then $y = -2x + 10$

For $3 \leq x < 4$, then $y = 4$.

For $4 \leq x$, then $y = 4x - 12$.

(d)



(e) $y = |x - 1| + |x - 3| + 2|x - 4| + 3|x - 5|$

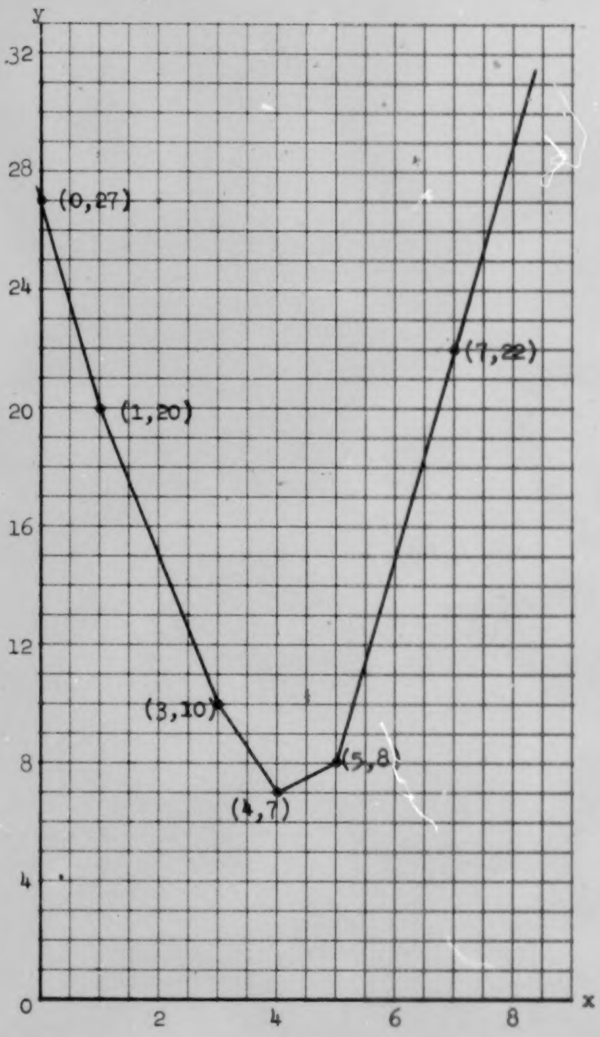
For $x < 1$, then $y = -7x + 27$.

For $1 \leq x < 3$, then $y = -5x + 25$.

For $3 \leq x < 4$, then $y = -3x + 19$.

For $4 \leq x < 5$, then $y = x + 3$.

For $5 \leq x$, then $y = 7x - 27$.



6. (a) Show that if $a > b > 0$, then $\frac{ab}{a+b} < b$.

$$\begin{aligned} 0 < b &\Rightarrow a < a + b \\ &\Rightarrow ab < b(a + b) \\ &\Rightarrow \frac{ab}{a+b} < b, \quad \text{since } a + b > 0. \end{aligned}$$

(b) Thus, show that for positive numbers a and b, the condition $\delta \leq \min(a,b)$ is satisfied by $\delta = \frac{ab}{a+b}$.

For $a \neq b$ the result follows from part (a). For $a = b$, $\delta = \frac{a}{2} < a$.

7. (a) Show for positive a, b that $\frac{a+b}{2} < \max(a,b)$ if $a \neq b$.

$$\frac{a+b}{2} < \frac{\max(a,b) + \max(a,b)}{2} \leq \max(a,b)$$

(b) Prove for all a, b that

$$\max(a,b) = \frac{1}{2}(a + b + |a - b|)$$

(c) $$\min(a,b) = \frac{1}{2}(a + b - |a - b|)$$

Assume, without loss of generality, that $a \geq b$,

then $\max(a,b) = a = \frac{1}{2}(a + b + a - b)$,

and $\min(a,b) = b = \frac{1}{2}(a + b - (a-b))$.

8. Show that

$$\max(a,b) + \max(c,d) \geq \max(a + c, b + d).$$

From Number 7(b)

$$\begin{aligned} \max(a,b) + \max(c,d) &= \tfrac{1}{2}(a + b + c + d + |a - b| + |c - d|), \\ \max(a + c, b + d) &= \tfrac{1}{2}(a + b + c + d + |a + c - (b + d)|) \\ &= \tfrac{1}{2}(a + b + c + d + |(a - b) + (c - d)|. \end{aligned}$$

The result follows at once.

9. Show that if $ab \geq 0$, then $ab \geq \min\{a^2, b^2\}$.

$$ab = |a|\ |b| \geq (\min\{|a|, |b|\})^2 = \min\{a^2, b^2\}.$$

10. Show that if $a = \max\{a, b, c\}$, then $-a = \min\{-a, -b, -c\}$.

If $a = \max\{a, b, c\}$, $a \geq b$, $a \geq c$,

then $-a \leq -b$ and $-a \leq -c$.

So, $-a = \min\{-a, -b, -c\}$.

11. Denote $\min\left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}\right\}$ by $\min\limits_r\left\{\frac{a_r}{b_r}\right\}$ and similarly for max.

If $b_r > 0$, $r = 1, 2, \ldots, n$, prove that

$$\min_r\left\{\frac{a_r}{b_r}\right\} \leq \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + b_n} \leq \max_r\left\{\frac{a_r}{b_r}\right\}.$$

Denote $\min\limits_r\left\{\frac{a_r}{b_r}\right\}$ by $\frac{a_k}{b_k}$, $1 \leq k \leq n$, and

$\max\limits_r\left\{\frac{a_r}{b_r}\right\}$ by $\frac{a_e}{b_e}$, $1 \leq e \leq n$.

Then, $\frac{a_k}{b_k} \leq \frac{a_r}{b_r}$, $r = 1, 2, \ldots, n$.

Or, $a_k b_r \leq b_k a_r$, for all r. Adding,

$$a_k b_1 + a_k b_2 + \ldots + a_k b_n \leq b_k a_1 + b_k a_2 + \ldots + b_k a_n.$$

Factoring and dividing, we obtain

$$\min\left\{\frac{a_r}{b_r}\right\} = \frac{a_k}{b_k} \leq \frac{a_1 + a_2 + \ldots + a_n}{b_1 + b_2 + \ldots + a_n}.$$

In the same way the remaining inequality may be obtained.

12. Prove that

$$\frac{1}{n} \leq \frac{1 + 2 + \ldots + n}{(n)^2 + (n-1)^2 + \ldots + 2^2 + 1^2} \leq 1 \quad \text{for} \quad n = 1, 2, 3, \ldots, n.$$

Use the inequality obtained in Number 11, with

$$\frac{a_r}{b_r} = \frac{r}{r^2} \text{ or } \frac{1}{r}.$$

13. (a) Prove directly from the properties of order for $\epsilon > 0$ that if $-\epsilon \leq x \leq \epsilon$ then $|x| \leq \epsilon$. Conversely, if $|x| \leq \epsilon$ then $-\epsilon \leq x \leq \epsilon$.

Suppose $-\epsilon \leq x \leq \epsilon$. If $0 < x$, $|x| = x \leq \epsilon$. If $x < 0$, $|x| = -x$. But $-\epsilon \leq x$ implies $-x \leq \epsilon$. So, $-x = |x| \leq \epsilon$.

Conversely, suppose $|x| \leq \epsilon$. If $0 \leq x$, $|x| = x$, so $-\epsilon < 0 \leq x \leq \epsilon$, thus $-\epsilon \leq x \leq \epsilon$. Similarly for $x < 0$.

(b) Prove that if x is an element of an ordered field and if $|x| < \epsilon$ for all positive values ϵ, then $x = 0$.

If $x \neq 0$, take $\epsilon = |x|$. We then have the contradictory statements $|x| = |x|$ and $|x| < |x|$.

14. (a) Prove that $|ab| = |a| \cdot |b|$.

Just consider the three cases

$$ab > 0, \quad ab = 0, \quad ab < 0.$$

(b) Prove that $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, $b \neq 0$.

From part(a) we have

$$\left|\frac{a}{b}\right| = \left|a\left(\frac{1}{b}\right)\right| = |a|\left|\frac{1}{b}\right|$$

and

$$|b| \cdot \left|\frac{1}{b}\right| = \left|\frac{b}{b}\right| = |1| = 1.$$

Hence $$\left|\frac{1}{b}\right| = \frac{1}{|b|}.$$

Therefore $$|a|\left|\frac{1}{b}\right| = |a| \cdot \frac{1}{|b|} = \frac{|a|}{|b|}.$$

15. Prove that $|x - y| \leq |x| + |y|$.

In $|a + b| \leq |a| + |b|$, set $a = x$, $b = -y$.

16. Under what conditions do the equality signs hold for

$$\big||a| - |b|\big| \leq |a + b| \leq |a| + |b|??$$

Equality occurs only if $a = b = 0$.

17. If $0 < x < 1$, we can multiply both sides of the inequality $x < 1$ by x to obtain $x^2 < x$ (and, similarly, we can show that $x^3 < x^2$, $x^4 < x^3$, and so on). Use this result to show that if $0 < |x| < 1$, then $|x^2 + 2x| < 3|x|$.

$$|x^2 + 2x| \leq |x^2| + |2x| \leq |x| + |2x| = 3|x|,$$

$$(|x^2| = |x|^2 < |x| \text{ since } 0 < |x| < 1).$$

18. Prove the following inequalities

(a) $x + \frac{1}{x} \geq 2$, $x > 0$.

Since $(x - 1)^2 \geq 0$, we have

$$x^2 - 2x + 1 \geq 0 \quad \text{or} \quad x^2 + 1 \geq 2x.$$

Since $\frac{1}{x} > 0$, we obtain $x + \frac{1}{x} \geq 2$.

(b) $x + \frac{1}{x} \leq -2, \quad x < 0.$

$(x + 1)^2 \geq 0$. So $x^2 + 2x + 1 \geq 0$, or $x^2 + 1 \geq -2x$.
Since $x < 0$, $\frac{1}{x} < 0$, so $\frac{1}{x}(x^2 + 1) \leq \frac{1}{x}(-2x)$ or $x + \frac{1}{x} \leq -2$.

(c) $|x + \frac{1}{x}| \geq 2, \quad x \neq 0.$

From (a) we have $x + \frac{1}{x} \geq 2$ for $x > 0$

or $|x + \frac{1}{x}| \geq 2$ for $x > 0$.

From (b) we have $-(x + \frac{1}{x}) \geq 2$ for $x < 0$

or $-(x + \frac{1}{x}) = |x + \frac{1}{x}| \geq 2$ for $x < 0$.

Thus $|x + \frac{1}{x}| \geq 2$ for $x \neq 0$.

19. Prove: $x^2 \geq x|x|$ for all real x.

If $x \geq 0$, $x = |x|$, and $x^2 = x|x|$.
If $x < 0$, $x|x| < 0 < x^2$.

20. Show that if $|x - a| < \frac{|a|}{2}$, then $\frac{|a|}{2} < |x| < \frac{3|a|}{2}$ for all $a \neq 0$.

Using the inequalities of 13(a), we obtain

$$|x - a| < \frac{|a|}{2} \rightarrow -\frac{|a|}{2} < x - a < \frac{|a|}{2}$$
$$\rightarrow \frac{|a|}{2} < x < \frac{3|a|}{2}.$$

21. Prove for positive a and b, where $a \neq b$, that

$$\frac{|b - a|^2}{4(a + b)} < \frac{a + b}{2} - \sqrt{ab} < \frac{|b - a|^2}{8\sqrt{ab}} .$$

To avoid $\sqrt{\ }$, let $a = m^2$, $b = n^2$, then we have to show that

$$\frac{|m^2 - n^2|^2}{4(m^2 + n^2)} < \frac{m^2 + n^2 - 2mn}{2} < \frac{|m^2 - n^2|^2}{8mn}$$

or

$$\frac{(m + n)^2}{2(m^2 + n^2)} < 1 < \frac{(m + n)^2}{4mn} .$$

Both of these inequalities are equivalent to $(m - n)^2 > 0$.

By way of example we show one of the equivalences:

$$\begin{aligned}(m - n)^2 > 0 &\rightarrow m^2 + n^2 > 2mn \\ &\rightarrow 2(m^2 + n^2) > (m + n)^2 \\ &\rightarrow 1 > \frac{(m + n)^2}{2(m^2 + n^2)} .\end{aligned}$$

TC A6-2. Definition of Limit of a Function

In some texts, the idea of limit often is expressed in words like these: "If, as x gets closer and closer to a, the values of $f(x)$ tend to the value L, then we call L the limit of $f(x)$ as x approaches a." The difficulty with this formulation, apart from the vagueness of the words, "gets closer and closer to," "tend to," is that it suggests the false notion that if x_2 is closer to a than is x_1, then $f(x_2)$ is closer to L than is $f(x_1)$.

Example TC A6-2. Consider

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

We have $\lim_{x \to 0} f(x) = 0$. Let $x_1 = \frac{1}{2n\pi}$ and $x_2 = \frac{2}{\pi(1 + 4n)}$ (n, a non-zero integer). Then $|x_2| < |x_1|$ but $|f(x_2)| > |f(x_1)|$ since $f(x_2) = \frac{2}{\pi(1 + 4n)}$ and $f(x_1) = 0$.

The above description of limit gives no clear idea of just how to verify that L is the limit of f as x approaches a in any particular case. We are compelled to give a definition which yields a clear-cut method of verification.

Quite early in our discussion we refer to Appendix 1-4 for an explanation of open and closed intervals. These ideas are essential to the material in this and succeeding sections. In the exercises, substantial use is made of ideas relating to the order properties of real numbers and absolute value: the student is expected to apply basic inequality theorems. The objective is to develop computational facility with absolute value as a background for proving facts about limits. As a lead into Section 3-3 we feel that it would be informative for the student to be given some numerical values for ϵ and be required to determine a δ sufficient to control the error (see, for example, Exercises 3-2, No. 11).

Solutions Exercises A6-2

The theorems of Section A6-1 provide the basis for the following arguments. In the general application of the transitive property we use the strong inequality in the conclusion since a strong inequality appears at least once in the chain of reasoning. We also make extended use of the inequalities

$$\big||a| - |b|\big| \le |a + b| \le |a| + |b|.$$

1. Show that if $0 < |x - a| < 1$, then $|x + 2a| < 1 + 3|a|$.

If $a < |x - a| < 1$, then

$$\begin{aligned} |x + 2a| &= |(x - a) + 3a| \\ &\le |x - a| + |3a| \\ &\le |x - a| + 3|a| \\ &< 1 + 3|a|. \end{aligned}$$

2. Show that if $0 < |x - a| < 1$, then $|x^3 - a^3| < (3|a^2| + 3|a| + 1)|x - a|$.

If $0 < |x - a| < 1$, then

$$\begin{aligned} |x^3 - a^3| &= |(x - a)(x^2 + ax + a^2)| \\ &= |x - a| \cdot |\big((x - a) + a\big)^2 + a\big((x - a) + a\big) + a^2| \\ &= |x - a| \cdot |(x - a^2) + 3a(x - a) + 3a^2| \\ &\le |x - a| \cdot [(x - a)^2 + 3|a| \cdot |x - a| + 3a^2] \\ &< 1 + 3|a| + 3a^2. \end{aligned}$$

3. Show that if $0 < |x - 2| < 1$, then $\frac{1}{|x - 4|} < 1$.

Hint: If $|x - 4| > 1$, then $\frac{1}{|x - 4|} < 1$.

We have $|x - 4| = |(x - 2) - 2|$ whence

$$|-2| - |x - 2| \le |x - 4| \le |x - 2| + |-2|.$$

Thus, if $0 < |x - 2| < 1$,

$$2 - 1 < |x - 4| < 1 + 2$$

or

$$1 < |x - 4| < 3$$

and

$$\frac{1}{|x - 4|} < 1.$$

4. Show that if $|x - a| < \frac{|a|}{2}$, then $\frac{1}{x^2} < \frac{4}{a^2}$.

We have $|x| = |(x - a) + a|$, so that

$$|a| - |x - a| \leq |x| \leq |x - a| + |a|.$$

Thus, if $|x - a| < \frac{|a|}{2}$,

$$|a| - \frac{|a|}{2} < |x| < \frac{|a|}{2} + |a|$$

or

$$\frac{|a|}{2} < |x| < \frac{3|a|}{2}$$

whence

$$\frac{a^2}{4} < x^2 < \frac{9a^2}{4},$$

from which the result follows.

5. Show that if $0 < |x - 1| < 1$, then $|4x + 1| < 9$ and $\left|\frac{1}{x + 2}\right| < 1$.

If $0 < |x - 1| < 1$, we have

$$\begin{aligned} |4x + 1| &= |4(x - 1) + 5| \\ &\leq 4|x - 1| + 5 \\ &< 4 \cdot 1 + 5 \\ &\leq 9. \end{aligned}$$

Also, if $0 < |x - 1| < 1$, we have

$$\begin{aligned} |x + 2| &= |(x - 1) + 3| \\ &\geq 3 - |x - 1| \\ &> 3 - 1 \\ &\geq 2 \end{aligned}$$

whence

$$\left|\frac{1}{x + 2}\right| < \frac{1}{2} < 1.$$

6. Show that if $0 < |x - 2| < 1$, then $|x + 1| < 4$ and $\left|\frac{1}{x^2 + 2x + 4}\right| < 1$.

If $|x - 2| < 1$, then

$$\begin{aligned} |x + 1| &= |(x - 2) + 3| \\ &\leq |x - 2| + 3 \\ &< 4. \end{aligned}$$

Since $x^2 + 2x + 4 = \big((x - 2) + 2\big)^2 + 2\big((x - 2) + 2\big) + 4$

$$= (x - 2)^2 + 6(x - 2) + 12$$

$$\begin{aligned} |x^2 + 2x + 4| &\geq 12 - |(x - 2)^2 + 6(x - 2)| \\ &\geq 12 - [(x - 2)^2 + 6|x - 2|]. \end{aligned}$$

Thus, if $0 < |x - 2| < 1$,

$$\begin{aligned} |x^2 + 2x + 4| &> 12 - (1 + 6) \\ &\geq 5. \end{aligned}$$

Finally, if $0 < |x - 2| < 1$,, we have

$$\frac{1}{|x^2 + 2x + 4|} < \frac{1}{5} < 1.$$

7. Estimate how large $x^2 + 1$ can become if x is restricted to the open interval $-3 < x < 1$.

If $-3 < x < 1$ then $3 > -x > -1$ whence

$$|x| < 3$$

and $$x^2 < 9,$$

so that $$x^2 + 1 < 10.$$

8. Use inequality properties to find a positive number M such that $0 < |x - 1| < 3$ for all x and

(a) $|x^2 + 2x + 4| \leq M$

(b) $|3x^2 - 2x + 3| \leq M$

We are required to submit any positive number M satisfying the given inequalities. It is not necessary to find the smallest possible number M.

The problem is included here to give the student preparatory experiences for Section A6-3. Because of this, the strategy is more valuable to the student than the actual solution.

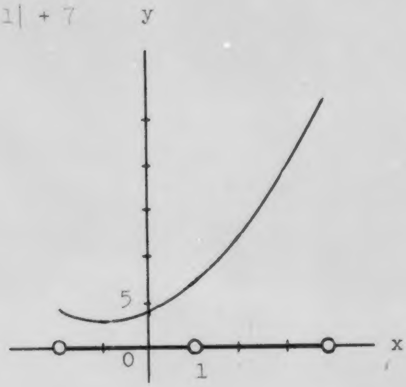
(a) $|x^2 + 2x + 4| \leq M$

For $0 < |x - 1| < 3$,

$$\begin{aligned} |x^2 + 2x + 4| &= |((x - 1) + 1)^2 + 2((x - 1) + 1) + 4| \\ &= |(x - 1)^2 + 4(x - 1) + 7| \\ &\leq (x - 1)^2 + 4|x - 1| + 7 \\ &< 3^2 + 4 \cdot 3 + 7 \\ &\leq 28. \end{aligned}$$

We may take M as any number, $M \geq 28$.

The graph $y = |x^2 + 2x + 4|$ shows that any number $M \geq 28$ will serve.



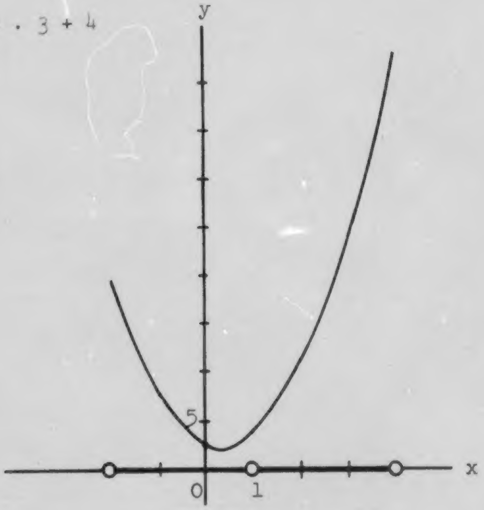
(b) $|3x^2 - 2x + 3| \leq M$

If $0 < |x - 1| < 3$, then

$$\begin{aligned} |3x^2 - 2x + 3| &= |3((x - 1) + 1)^2 - 2((x - 1) + 1) + 3| \\ &= |3(x - 1)^2 + 4(x - 1) + 4| \\ &\leq 3(x - 1)^2 + 4|x - 1| + 4 \\ &< 3 \cdot 3^2 + 4 \cdot 3 + 4 \\ &\leq 43. \end{aligned}$$

We take $M \geq 43$.

The graph of $y = |3x^2 - 2x + 3|$ shows that any number $M \geq 43$ will do.



9. (a) Show that if $0 < |x - 3| < 1$ and $0 < |x - 3| < \frac{\epsilon}{7}$, then $|x^2 - 9| < \epsilon$.

$$|x^2 - 9| = |(x - 3)\big((x - 3) + 6\big)|$$
$$\leq |x - 3| \cdot (|x - 3| + 6).$$

Thus, if $0 < |x - 3| < 1$ and $0 < |x - 3| < \frac{\epsilon}{7}$,

$$|x^2 - 9| < \frac{\epsilon}{7}(1 + 6)$$

or

$$|x^2 - 9| < \epsilon.$$

(b) Show that the pair of inequalities $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{7}$ (or $\delta \leq \min\{1, \frac{\epsilon}{7}\}$) is satisfied by $\delta = \frac{\epsilon}{7 + \epsilon}$.

For $\epsilon > 0$,

$$\frac{\epsilon}{7 + \epsilon} = \frac{(7 + \epsilon) - 7}{7 + \epsilon} = \frac{7 + \epsilon}{7 + \epsilon} - \frac{7}{7 + \epsilon}$$
$$= 1 - \frac{7}{7 + \epsilon}$$
$$< 1 \quad (\text{since } \frac{7}{7 + \epsilon} > 0).$$

Also, for $\epsilon > 0$,

$$\frac{\epsilon}{7 + \epsilon} = \epsilon \cdot \frac{1}{7 + \epsilon} < \epsilon \cdot \frac{1}{7} \quad (\text{since } \frac{1}{7 + \epsilon} < \frac{1}{7}).$$

Since $\frac{\epsilon}{7 + \epsilon} < \min\{1, \frac{\epsilon}{7}\}$, the result follows.

10. Find a number $M \geq 1$ such that $\left|\frac{x + 4}{x - 4}\right| \leq M$ for all x such that $0 < |x - 2| < 1$. (See No. 3 above.)

If $0 < |x - 2| < 1$ then $\frac{1}{|x - 4|} < 1$ from Number 3 and

$$|x + 4| = |(x - 2) + 6|$$
$$\leq |x - 2| + 6$$
$$< 1 + 6 \leq 7.$$

Thus, under these conditions,

$$\left|\frac{x + 4}{x - 4}\right| < |x + 4| < 7.$$

We take M as any number, $M \geq 7$.

11. For the given value of ϵ, find a number δ such that if $0 < |x - 3| < \delta$, $|x^2 - 9| < \epsilon$.

(a) $\epsilon = 0.1$

(b) $\epsilon = 0.01$

Is your choice of δ in (b) acceptable as an answer in (a)? Explain.

$$\begin{aligned} |x^2 - 9| &= |x - 3| \cdot |(x - 3) + 6| \\ &\leq |x - 3| \cdot (|x - 3| + 6) \\ &< \delta(\delta + 6). \end{aligned}$$

(At the last line we used $0 < |x - 3| < \delta$.) For convenience, we restrict δ so that $\delta \leq 1$. Then, under this condition, $|x^2 - 9| < 7\delta$.

(a) To insure that $|x^2 - 9| < 0.1$ we may take $\delta = \frac{0.1}{7} = \frac{1}{70}$.

(b) To insure that $|x^2 - 9| < 0.01$ we take $\delta = \frac{0.01}{7} = \frac{1}{700}$.

The choice, $\delta = \frac{1}{700}$, is acceptable in (a), for if $0 < |x - 3| < \frac{1}{700}$ then

$$|x^2 - 9| < 7\delta \leq 0.01 < 0.1.$$

12. For the following functions, find the limit L as x approaches a. For each value of ϵ, exhibit a number δ such that $|f(x) - L| < \epsilon$ whenever $|x - a| < \delta$.

(a) $f(x) = 3x - 2$, $a = \frac{1}{2}$.

(b) $f(x) = mx + b$, $(m \neq 0)$.

(c) $f(x) = 1 + x^2$, $a = 0$.

(a) $\lim\limits_{x \sim \frac{1}{2}} (3x - 2) = -\frac{1}{2}$.

We have

$$\begin{aligned} |(3x - 2) - (-\tfrac{1}{2})| &= |3x - \tfrac{3}{2}| \\ &= 3|x - \tfrac{1}{2}|. \end{aligned}$$

We wish to find a δ such that whenever $|x - \frac{1}{2}| < \delta$ then

$$|(3x - 2) = (-\tfrac{1}{2})| < \epsilon.$$

We take $\delta = \frac{\epsilon}{3}$. Then if $|x - \frac{1}{2}| < \delta$,

$$\begin{aligned} |(3x - \tfrac{1}{2}) - \tfrac{1}{2}| &= 3|x - \tfrac{1}{2}| \\ &< 3\delta \\ &\leq \epsilon. \end{aligned}$$

(b) $f(x) = mx + b$, $(m \neq 0)$.

$$\lim_{x \to a} f(x) = ma + b$$

$$\begin{aligned} |(mx + b) - (ma + b)| &= |m(x - a)| \\ &= |m| \cdot |x - a|. \end{aligned}$$

We wish to find a δ such that whenever $|x - a| < \delta$ then

$$|m| \cdot |x - a| < \epsilon.$$

We take $\delta = \frac{\epsilon}{|m|}$. For this choice of δ, whenever $|x - a| < \delta$,

$$\begin{aligned} |(mx + b) - (ma + b)| &= |m| \cdot |x - a| \\ &< |m| \cdot \delta \\ &\leq \epsilon. \end{aligned}$$

(c) $f(x) = 1 + x^2$, $a = 0$.

$$\lim_{x \to 0} (1 + x^2) = 1$$

$$|(1 + x^2) - 1| = x^2.$$

We wish to find a $\delta > 0$ such that $|(1 + x^2) - 1| < \epsilon$ whenever $|x - 0| < \delta$. We take $\delta = \sqrt{\epsilon}$. For this choice of δ, if $|x - 0| < \delta$,

$$\begin{aligned} |(1 + x^2) - 1| &= x^2 \\ &< \delta^2 \\ &\leq \epsilon. \end{aligned}$$

TC A6-3. Epsilonic Technique

The importance of technical mastery is lost on some students, usually among the brightest. It may be necessary to emphasize for them the connection between mechanical skills and a conceptual grasp of the subject. Just as an accomplished musician can perceive the essence of a composition without stumbling over individual notes, the accomplished user of mathematics must have enough mechanical facility to be above distraction by mechanical details.

A great deal of pedagogical consideration has gone into the composition of Section A6-3. The student should develop an operationally satisfactory way of working with the idea of limit. Memorization of Definition A6-2 is certainly not sufficient. Nevertheless, definitions are like the fixed stars. They give the student a firm criterion for knowing where he is.

We wish to cultivate the attitude of inquiry in which the student asks himself the following questions:

1. Do I have suitable approximations for L? (The answer should be easy since the approximations are usually taken at endpoints or at interior points of a defined interval.)
2. Do I have a candidate for L? If so, what is it?
3. How shall I test the candidate to see if it is the limit? Can I keep the error within any given tolerance ϵ by confining the points x to a suitable δ-neighborhood of a?

It is easy to show that if, for an arbitrary $\epsilon > 0$,, there exists a control, $\delta > 0$, then any smaller positive number δ^* will certainly suffice for the same ϵ. For, let there exist a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$. Further, let δ^* be any number, $0 < \delta^* < \delta$. It follows at once that for all x such that $0 < |x - a| < \delta^*$ we have $0 < |x - a| < \delta$; whence, for all these x, $|f(x) - L| < \epsilon$.

We are not concerned with the largest δ that gives the desired degree of control over the error tolerance ϵ; rather, we seek any number δ which is sufficient. The task is often simplified if we agree to restrict δ to numbers less than 1. This restriction simply means that we are focusing our attention on the deleted interval $\{x : 0 < |x - a| < \delta < 1\}$..

We noted in Section A6-3 (Step 1. Simplification) that frequently we are able to find a simple function $g : \delta \to c\delta$, c a positive constant. This is the case because we are dealing primarily with functions which have continuous derivatives on a full neighborhood of a. From the fact that f is continuous, we have

$$L = \lim_{x \to a} f(x) = f(a).$$

From the fact that f' is continuous on a neighborhood we know that $|f'(x)|$ is also continuous on any closed interval centered at a within the neighborhood (composition of continuous functions Section 3-6). Consequently, $|f'(x)|$ has a maximum value on the interval. (Extreme Value Theorem, Section 3-7). Let K be any value greater than the maximum, so that $|f'(x)| < K$ on the interval. Now, assuming $0 < |x - a| < \delta$, where this deleted δ-neighborhood lies within the interval there exists a value ξ within the δ-neighborhood (Law of the Mean, Chapter 3) such that

$$\begin{aligned} |f(x) - L| &= |f(x) - f(a)| \\ &= |f'(\xi)(x - a)| \\ &= |f'(\xi)| \cdot |x - a| \\ &< K|x - a| \\ &< K\,\delta. \end{aligned}$$

The method of bounding the denominator in Example 3-3e is given because it is a routine procedure conforming to the letter of our general outline. A short cut is to anticipate the problem of bounding x away from 0 at line (1). We may recognize at once that, since $|x - a| < \delta$, the distance from x to a is no larger than δ; we may then keep x away from 0 by requiring δ to be less than the distance $|a|$ of a from 0. To achieve this we may take $\delta \le \frac{|a|}{2}$.

For your reference we list the following generalities:

1. The definition of limit employs only values of x different from a.
2. Limit is a local property (sometimes called a property in the small) involving the behavior of a function within any (deleted) neighborhood of a point.

3. The existence of the limit of f at a point implies that f is defined for some values of x in every deleted neighborhood of a; that is, $f(x)$ exists for some values of x arbitrarily near a.

4. The limit is independent of the choice of the deleted neighborhood of a.

5. The assertion that the function f has the limit L as x approaches a is not the same as saying $f(a) = L$, nor is it the same as saying that L is an upper (lower) bound of $f(x)$.

6. The value of δ depends upon the value of ϵ (exception: $f(x) = c$, c constant).

A careful distinction should be made between the analysis of a problem and its exposition. This is particularly necessary in the case of limit proofs. Steps 1 and 2 show how the solution is found: in Step 1 we examine the problem and set up a simplified model; in Step 2 we outline our plan of attack or strategy. Step 3 is the actual proof where it is verified that the solution has been found. An attempt should be made to develop elegance of style in presenting proofs.

Solutions Exercises A6-3

In the following epsilonic arguments the analysis (Steps 1 and 2 in the pattern of the text) precedes the proof. We make liberal use of the inequalities

$$\big||a| - |b|\big| \le |a + b| \le |a| + |b|$$

(Section A6-1). In the selection on δ (in Numbers 4b - 4g) it is expedient to restrict δ by the auxiliary conditions that $\delta \le 1$. The proof (verification) is simplified by an application of Exercises A6-3a, Number 5(b).

1. Prove $\lim\limits_{x \to 4} (\frac{1}{2}x - 3) = -1$: obtain an upper bound $g(\delta)$ for the absolute error and find δ in terms of ϵ.

   In this problem we write out the steps in detail.

   To prove that $\lim\limits_{x \to 4} (\frac{1}{2}x - 3) = -1$.

   For each $\epsilon > 0$ obtain a δ.
   Show: if $0 < |x - 4| < \delta$, then $|(\frac{1}{2}x - 3) - (-1)| < \epsilon$.

   Step 1.

   (a)
   $$\begin{aligned} |(\tfrac{1}{2}x - 3) - (-1)| &= |\tfrac{1}{2}x - 2| \\ &= \tfrac{1}{2}|x - 4|. \end{aligned}$$

(b) If $0 < |x - 4| < \delta$,

$$\left|\left(\frac{1}{2}x - 3\right) - (-1)\right| = \frac{1}{2}|x - 4|$$
$$< \frac{1}{2}\delta.$$

(c) Take $g(\delta) = \frac{1}{2}\delta$.

Step 2. To make $g(\delta) \leq \epsilon$, set $\delta = 2\epsilon$.

Step 3. If $\delta = 2\epsilon$ and $0 < |x - 4| < \delta$, then

$$\left|\left(\frac{1}{2}x - 3\right) - (-1)\right| = \frac{1}{2}|x - 4|$$
$$< \frac{1}{2}(2\epsilon)$$
$$\leq \epsilon.$$

2. Give arguments that prove

   (a) $\lim_{x \to a} c = c$, c any constant.

   (b) $\lim_{x \to a} x = a$.

   (c) $\lim_{x \to a} kx = ka$, k any constant.

   (Use the results of Example 3-3a of the text for parts b and c.)

(a) $\lim_{x \to a} c = c$, c constant.

Statement of problem:

For each $\epsilon > 0$ we obtain a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|c - c| < \epsilon$.

Since $|c - c| = 0$ is less than $\epsilon > 0$ for all δ we arbitrarily take any positive number for δ, say $\delta = 1$.

Then for $\delta = 1$,, whenever $0 < |x - a| < \delta$, we have $|c - c| < \epsilon$.

(b) $\lim_{x \to a} x = a$.

From Example A6-3a we have

$$\lim_{x \to a} (mx + b) = ma + b, \quad m \neq 0,$$

whence for $m = 1$, $b = 0$,

$$\lim_{x \to a} x = a.$$

(c) $\lim_{x \to a} kx = ka$, k constant.

If $k \neq 0$, the result is a direct consequence of Example 3-3a; if $k = 0$, the result follows from part (a).

3. Invoke the definition directly to prove the existence of the limits to Problem 2.

(a) See answer to Number 2(a).

(b) $\lim_{x \to a} x = a$..

We follow the pattern of Example A6-3a in Step 1. Then take $g(\delta) = \delta$.

To make $g(\delta) \leq \epsilon$ we take $\delta = \epsilon$.

Thus, if $\delta = \epsilon$ and $0 < |x - a| < \delta$, then $|x - a| < \delta \leq \epsilon$.

(c) $\lim_{x \to a} kx = ka$, k constant.

We follow the pattern of Example A6-3a through Step 2 and take $\delta = \frac{\epsilon}{|m|}$ where $m = k$.

For $\delta = \frac{\epsilon}{|k|}$ and $0 < |x - a| < \delta$,

$$\begin{aligned} |kx - ka| &= |k| \cdot |x - a| \\ &< |k|\delta \\ &\leq \epsilon. \end{aligned}$$

4. In each of the following guess the limit, and then prove that your guess is correct: give an expression for $g(\delta)$ and find δ in terms of ϵ.

(a) $\lim_{x \to 0} \frac{1}{1 + x^2}$

(b) $\lim_{x \to 3} \frac{x^2(x - 3)}{x - 3}$

(c) $\lim_{x \to a} \frac{x^3 - a^3}{x - a}$

(d) $\lim_{x \to 1} \frac{x + 1}{x^2 + 1}$

(e) $\lim_{x \to 2} \frac{x^2 - 4}{x^3 - 8}$

(f) $\lim_{x \to 0} \frac{x^3 - 3x - 1}{x + 2}$

(g) $\lim_{x \to 1} \frac{4x^2 - 3x - 1}{x + 2}$

We omit repetitious material. The statement of the problem follows the pattern in the text.

(a) $\lim\limits_{x \to 0} \dfrac{1}{1 + x^2} = 1.$

$$\left|\frac{1}{1 + x^2} - 1\right| = \left|\frac{x^2}{1 + x^2}\right| = \frac{x^2}{1 + x^2}$$

$$< x^2 \qquad \text{(since } 1 + x^2 \geq 0 \text{)}$$

$$< \delta^2 \qquad \text{if } 0 < |x| < \delta.$$

Take $g(\delta) = \delta^2$ and set $\delta = \sqrt{\epsilon}$.

Verification:

If $\delta = \sqrt{\epsilon}$ and $0 < |x| < \delta$, then $\left|\dfrac{1}{1 + x^2} - 1\right| < \delta^2 \leq \epsilon$.

(b) $\lim\limits_{x \to 3} \dfrac{x^2(x - 3)}{x - 3} = 9$.

$$\left|\frac{x^2(x - 3)}{x - 3} - 9\right| = |x^2 - 9| \qquad \text{for } x \neq 3$$

$$= |x - 3| \cdot |(x - 3) + 6|$$

$$\leq |x - 3| \cdot (|x - 3| + 6)$$

$$< \delta(\delta + 6).$$

(At the last line we used $0 < |x - 3| < \delta$.)

For convenience we restrict δ by requiring that $\delta \leq 1$. Under this condition if $0 < |x - 3| < \delta$

$$\left|\frac{x^2(x - 3)}{x - 3} - 9\right| < 7\delta.$$

Take $g(\delta) = 7\delta$. Obtain a δ satisfying the two conditions $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{7}$, simultaneously. One way to do this is to take $\delta = \frac{\epsilon}{7 + \epsilon}$ (Exercises A6-2, No. 9(b)).

Verification: If $\delta = \frac{\epsilon}{7 + \epsilon}$ and $0 < |x - 3| < \delta$

$$\left|\frac{x^2(x - 3)}{x - 3} - 9\right| < 7\delta$$

$$\leq 7 \cdot \frac{\epsilon}{7 + \epsilon}$$

$$\leq \frac{7}{7 + \epsilon} \cdot \epsilon$$

$$< \epsilon \qquad \left(\text{since } \frac{7}{7 + \epsilon} < 1\right).$$

(c) $\lim\limits_{x \to a} \dfrac{x^3 - a^3}{x - a} = 3a^2$.

We have $\dfrac{x^3 - a^3}{x - a} = x^2 + ax + a^2$ for $x \neq a$,, whence

$$\begin{aligned}
\left|\frac{x^3 - a^3}{x - a} - 3a^2\right| &= |x^2 + ax - 2a^2| \\
&= \left|\big((x - a) + a\big)^2 + a\big((x - a) + a\big) - 2a^2\right| \\
&= |(x - a)^2 + 3a(x - a)| \\
&= |x - a| \cdot |(x - a) + 3a| \\
&\leq |x - a| \cdot (|x - a| + 3|a|) \\
&< \delta(\delta + 3|a|)
\end{aligned}$$

(At the last line we used $0 < |x - a| < \delta$.)

For convenience we restrict δ by requiring $\delta \leq 1$. Under this condition if $0 < |x - a| < \delta$,

$$\left|\frac{x^3 - a^3}{x - a} - 3a^2\right| < \delta(1 + 3|a|).$$

We choose δ to satisfy the conditions $\delta \leq 1$ and $\delta \leq \dfrac{\epsilon}{1 + 3|a|}$; i.e., $\delta \leq \min\{1, \dfrac{\epsilon}{1 + 3|a|}\}$. A convenient way to satisfy these conditions is to take $\delta = \dfrac{\epsilon}{1 + 3|a| + \epsilon}$ (Exercises A6-1, No. 6(b)).

Verification:

If $\delta = \dfrac{\epsilon}{1 + 3|a| + \epsilon}$ and $0 < |x - a| < \delta$,, then

$$\begin{aligned}
\left|\frac{x^3 - a^3}{x - a} - 3a^2\right| &= |x - a| \cdot |(x - a) + 3a| \\
&\leq |x - a| \cdot (|x - a| + 3|a|) \\
&< \delta(1 + 3|a|) \\
&\leq \frac{\epsilon}{1 + 3|a| + \epsilon} \cdot (1 + 3|a|) \\
&\leq \epsilon \cdot \frac{1 + 3|a|}{1 + 3|a| + \epsilon} \\
&< \epsilon \qquad \left(\text{since } \frac{1 + 3|a|}{1 + 3|a| + \epsilon} < 1\right).
\end{aligned}$$

(d) $\lim\limits_{x \to 1} \dfrac{x + 1}{x^2 + 1} = 1.$

$$\left|\frac{x + 1}{x^2 + 1} - 1\right| = \left|\frac{x - x^2}{x^2 + 1}\right| = \frac{|x - 1| \cdot |x|}{1 + x^2}$$

$$\begin{aligned}
&\le |x - 1| \cdot |x| && (\text{since } 1 + x^2 \ge 1)\\
&\le |x - 1| \cdot |(x - 1) + 1|\\
&\le |x - 1| \cdot (|x - 1| + 1)\\
&< \delta(\delta + 1) && \text{if } |x - 1| < \delta.
\end{aligned}$$

For convenience we restrict δ by requiring $\delta \le 1$. Under this condition if $|x - 1| < \delta$

$$\begin{aligned}
\left|\frac{x + 1}{x^2 + 1} - 1\right| &< \delta(\delta + 1)\\
&\le 2\delta.
\end{aligned}$$

To satisfy the two conditions $\delta \le 1$ and $\delta \le \frac{\epsilon}{2}$ simultaneously we take (for convenience) $\delta = \dfrac{\epsilon}{2 + \epsilon}$.

Verification:

If $\delta = \dfrac{\epsilon}{2 + \epsilon}$ and $|x - 1| < \delta$,

$$\begin{aligned}
\left|\frac{x + 1}{x^2 + 1} - 1\right| &< 2\delta\\
&\le 2 \cdot \frac{\epsilon}{2 + \epsilon}\\
&\le \frac{2}{2 + \epsilon} \cdot \epsilon\\
&< \epsilon && \left(\text{since } \frac{2}{2 + \epsilon} < 1\right).
\end{aligned}$$

(e) $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x^3 - 8} = \dfrac{1}{3}.$

$$\begin{aligned}
\left|\frac{x^2 - 4}{x^3 - 8} - \frac{1}{3}\right| &= \left|\frac{x + 2}{x^2 + 2x + 4} - \frac{1}{3}\right| && (x \ne 2)\\
&= \left|\frac{-x^2 + x + 2}{3(x^2 + 2x + 4)}\right|\\
&= \frac{1}{3}\left|\frac{(x - 2)(x + 1)}{x^2 + 2x + 4}\right|.
\end{aligned}$$

For convenience we restrict δ by requiring $\delta \leq 1$. **Under this** condition, if $0 < |x - 2| < \delta$,

$$\begin{aligned}\left|\frac{x^2 - 4}{x^3 - 8} - \frac{1}{3}\right| &< \frac{1}{3}\,|(x - 2)(x + 1)| \qquad \text{(from Exercises A6-2, No. 6)}\\ &\leq \frac{1}{3}\,|x - 2| \cdot |(x - 2) + 3|\\ &\leq \frac{1}{3}\,|x - 2| \cdot (|x - 2| + 3)\\ &< \frac{1}{3}\,\delta(1 + 3)\\ &\leq \frac{4}{3}\,\delta.\end{aligned}$$

We wish to obtain a value δ satisfying the two conditions $\delta \leq 1$ and $\delta \leq \frac{3}{4}\epsilon$, simultaneously. One way to satisfy the conditions is to set

$$\delta = \frac{\frac{3}{4}\epsilon}{1 + \frac{3}{4}\epsilon} = \frac{3\epsilon}{4 + 3\epsilon} \qquad \text{(Exercises A6-1, No. 6(b)).}$$

Verification:

If $\delta = \frac{3\epsilon}{4 + 3\epsilon}$ and $0 < |x - 2| < \delta$,

$$\begin{aligned}\left|\frac{x^2 - 4}{x^3 - 8} - \frac{1}{3}\right| &\leq \frac{1}{3}\,|x - 2| \cdot (|x - 2| + 3)\\ &< \frac{4}{3}\,\delta\\ &\leq \frac{4}{3} \cdot \frac{3\epsilon}{4 + 3\epsilon}\\ &\leq \frac{12}{3(4 + 3\epsilon)} \cdot \epsilon\\ &< \epsilon \qquad \left(\text{since } \frac{12}{3(4 + 3\epsilon)} < 1\right).\end{aligned}$$

(f) $\lim\limits_{x \sim 0} \frac{x^3 - 3x - 1}{x + 2} = -\frac{1}{2}$.

$$\left|\frac{x^3 - 3x - 1}{x + 2} - \left(-\frac{1}{2}\right)\right| = \left|\frac{2x^3 - 5x}{2(x + 2)}\right|.$$

For convenience we rectrict δ by requiring $\delta \leq 1$. **Thus if** $0 < |x| < \delta \leq 1$, $|x + 2| \geq 2 - |x| \geq 1$ and $\frac{1}{|x + 2|} \leq 1$.

Thus if $\delta \leq 1$ and $0 < |x| < \delta$, we have

$$\begin{aligned}
\left|\frac{x^3 - 3x - 1}{x + 2} - \left(-\frac{1}{2}\right)\right| &= \left|\frac{2x^3 - 5x}{2(x + 2)}\right| \\
&\leq \frac{1}{2}\, |2x^3 - 5x| \\
&\leq \frac{1}{2} \cdot |x| \cdot |2x^2 - 5| \\
&\leq |x| \cdot \left(x^2 + \frac{5}{2}\right) \\
&< \delta \cdot \left(1 + \frac{5}{2}\right) \\
&< 4\delta \qquad \left(\text{since } \frac{7}{2}\,\delta < 4\delta\right).
\end{aligned}$$

To satisfy the two conditions $\delta \leq 1$ and $\delta \leq \frac{\epsilon}{4}$ simultaneously, we take, for convenience, $\delta = \frac{\epsilon}{4 + \epsilon}$.

Verification:
Set $\delta = \frac{\epsilon}{4 + \epsilon}$ in the statement of the problem, as in (a) - (e). In the last step we have, under the conditions that $0 < |x - 0| < \delta$,

$$\begin{aligned}
\left|\frac{x^3 - 3x - 1}{x + 2} - \left(-\frac{1}{2}\right)\right| &< 4\delta \\
&\leq 4 \cdot \frac{\epsilon}{4 + \epsilon} \\
&< \epsilon \qquad \left(\text{since } \frac{4}{4 + \epsilon} < 1\right).
\end{aligned}$$

(g) $\lim\limits_{x \to 1} \dfrac{4x^2 - 3x - 1}{x + 2} = 0$.

$$\begin{aligned}
\left|\frac{4x^2 - 3x - 1}{x + 2}\right| &= \frac{|(x - 1)(4x + 1)|}{|x + 2|} \\
&= \frac{|x - 1| \cdot |4(x - 1) + 5|}{|x + 2|}.
\end{aligned}$$

We restrict δ so that $\delta \leq 1$. Under this restriction, if $0 < |x - 1| < \delta$, we have

$$\begin{aligned}
\left|\frac{4x^2 - 3x - 1}{x + 2}\right| &< |x - 1| \cdot |4(x - 1) + 5| \qquad \text{(Exercises A6-2, No. 5)} \\
&\leq |x - 1| \cdot (4|x - 1| + 5) \\
&< \delta(4\delta + 5) \\
&\leq 9\delta.
\end{aligned}$$

We require that $\delta \leq \min(1, \frac{\epsilon}{9})$. This condition is satisfied if we take $\delta = \frac{\epsilon}{9 + \epsilon}$.

Verification:

If $\delta = \frac{\epsilon}{9 + \epsilon}$ and $0 < |x - 1| < \delta$, then

$$\begin{aligned}
\left|\frac{4x^2 - 3x - 1}{x + 2}\right| &< |(x - 1)(4x + 1)| \\
&\leq |x - 1| \cdot (4|x - 1| + 5) \\
&< 9\delta \\
&\leq 9 \cdot \frac{\epsilon}{9 + \epsilon} \\
&< \epsilon \qquad \text{(since } \frac{\epsilon}{9 + \epsilon} < 1\text{).}
\end{aligned}$$

TC A6-4. Limit Theorems

Theorems become tools to the student only in so far as he understands the hypotheses and appreciates conclusions that derive therefrom. Graphical considerations are usually helpful in visualizing abstract properties and proofs (in particular, Theorems A6-4a, b, c, f). Proofs of the theorems should be made plausible, but memorization of formal proofs is inconsistant with the philosophy of this course.

The statement $\lim\limits_{x \sim a} f(x) = L$ simply means that L satisfies the conditions of Definition A6-2. It is conceivable that another number M might satisfy the same conditions. Then the statements $\lim\limits_{x \sim a} f(x) = L$ and $\lim\limits_{x \sim a} f(x) = M$ would both be true. If this could happen, then $\lim\limits_{x \sim a} f(x)$ could have no meaning by itself. Consequently, each limit theorem would need to be appropriately interpreted; for example, Theorem A6-4c would be stated:

THEOREM A6-4c. If $\lim\limits_{x \sim a} f(x) = L$ and $\lim\limits_{x \sim a} g(x) = M$, then $L + M$ is a limit of $f + g$ at $x = a$.

This interpretation is unnecessary, since $\lim\limits_{x \sim a} f(x)$, if it exists, is unique.

THEOREM. If L and M are limits of f at a, that is, if $\lim\limits_{x \sim a} f(x) = L$ and $\lim\limits_{x \sim a} f(x) = M$, then $L = M$.

Proof. Suppose $L > M$, take $\epsilon = \frac{L - M}{3} > 0$, then there are $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$-\epsilon < L - f(x) < \epsilon \qquad \text{for } 0 < |x - a| < \delta_1,$$

$$-\epsilon < f(x) - M < \epsilon \qquad \text{for } 0 < |x - a| < \delta_2.$$

Hence both inequalities hold for

$$0 < |x - a| < \min(\delta_1, \delta_2).$$

Thus $0 < L - M < 2\epsilon$. Impossible

Sometimes assumptions are left tacit. For example, in the proof of Theorem A6-4c, ϵ^* is, of course, subject to the conditions imposed upon ϵ, that is, $\epsilon^* > 0$.

In Theorem A6-4c, there is a tolerance, ϵ, for the sum $(f + g)$ and tolerances, ϵ_1 and ϵ_2, for the addends (f and g,, respectively). It is sufficient that these tolerances satisfy the condition $\epsilon_1 + \epsilon_2 \leq \epsilon$: for convenience (and for definiteness) we take $\epsilon_1 = \epsilon_2 = \frac{\epsilon}{2}$, in the text. The control δ_1 maintains the tolerance $\epsilon_1 = \frac{\epsilon}{2}$ for f and control δ_2 maintains $\epsilon_2 = \frac{\epsilon}{2}$ for g. It follows that $\delta \leq \min(\delta_1, \delta_2)$ will maintain the tolerance $\frac{\epsilon}{2}$ for f and also for g.

We can obtain a slightly different proof of Theorem A6-4d by utilizing Lemma A6-4 in the following way. By hypothesis, corresponding to any positive ϵ_1, ϵ_2 we can find δ_1, δ_2 such that

$$|f(x) - L| < \epsilon_1 \quad \text{whenever} \quad 0 < |x - a| < \delta_1,$$

$$|g(x) - M| < \epsilon_2 \quad \text{whenever} \quad 0 < |x - a| < \delta_2,$$

and by Lemma A6-4 there is a δ^* such that

$$|g(x)| < \frac{3}{2}|M| \quad \text{whenever} \quad 0 < |x - a| < \delta^*.$$

Since $f(x)g(x) - LM = \big(f(x) - L\big)g(x) + L\big(g(x) - M\big)$, if $|f(x) - L| < \epsilon_1$, $|g(x) - M| < \epsilon_2$, and $|g(x)| < \frac{3}{2}|M|$ then

$$\begin{aligned} |f(x)g(x) - LM| &\leq |f(x) - L| \cdot |g(x)| + |L| \cdot |g(x) - M| \\ &< \epsilon_1 \frac{3}{2}|M| + |L|\,\epsilon_2. \end{aligned}$$

In order to remain within the tolerance ϵ we take $\epsilon_1 = \epsilon_2$ and

$$\epsilon_1 \frac{3}{2}|M| + |L|\epsilon_2 = \epsilon_1\left(\frac{3}{2}|M| + |L|\right) \leq \epsilon.$$

Then to cover the case where $M = L = 0$ we take

$$\epsilon_1 = \epsilon_2 = \frac{\epsilon}{\frac{3}{2}|M| + |L| + 1}$$

Let δ_1, δ_2, and δ^* be appropriate controls for this choice of ϵ_1 and ϵ_2 and take

$$\delta = \min(\delta_1, \delta_2, \delta^*).$$

For this choice of δ and for

$$0 < |x - a| < \delta$$

we have

$$\begin{aligned} |f(x)g(x) - LM| &\le |f(x) - L| \cdot |g(x)| + |L| \cdot |g(x) - M| \\ &\le \frac{\epsilon}{\frac{3}{2}|M| + |L| + 1} \cdot \left(\frac{3}{2}|M| + |L|\right) \\ &< \epsilon. \end{aligned}$$

Sum notation may be used to express a linear combination

$$\phi(x) = \sum_{i=1}^{n} c_i f_i(x)$$

and state the corollary to Theorem A6-4c,

$$\begin{aligned} \lim_{x \to a} \sum_{i=1}^{n} c_i f_i(x) &= \sum_{i=1}^{n} c_i \lim_{x \to a} f_i(x) \\ &= \sum_{i=1}^{n} c_i L_i . \end{aligned}$$

Mathematical induction (Appendix 3) is required for the proof of the corollaries to Theorems A6-4c and A6-4d, respectively. If the student works through these proofs he will gain a deeper understanding of the fundamental results and an appreciation of the power of mathematical induction.

In Lemma A6-4 we have "... a δ-neighborhood of a wherein $g(x)$ is closer to M than to zero." This means: for all x in the δ-neighborhood and in the domain of g, $g(x)$ is closer to M than to zero.

SQUEEZE THEOREM (Proof). Let I be the deleted neighborhood of a where

$$h(x) \le f(x) \le g(x).$$

For every $\epsilon > 0$, I contains a deleted δ-neighborhood of a wherein $|h(x) - M| < \epsilon$ and $|g(x) - M| < \epsilon$. Equivalently, we have

$$M - \epsilon < h(x), \quad g(x) < M + \epsilon$$

whenever $0 < |x - a| < \delta$. It follows that

$$M - \epsilon < f(x) < M + \epsilon$$

or

$$|f(x) - M| < \epsilon$$

whenever $0 < |x - a| < \delta$.

Solutions Exercises A6-4

1. Prove the corollary to Theorem A6-4c. **The limit of a linear combination of** functions is the same linear combination of the limits of the functions; i.e., if $\lim\limits_{x \to a} f_i(x) = L_i$, $i = 1, 2, \ldots, n$, then

$$\lim_{x \to a} \sum_{i=1}^{n} c_i f_i(x) = \sum_{i=1}^{n} \lim_{x \to a} c_i f_i(x)$$

$$= \sum_{i=1}^{n} c_i \lim_{x \to a} f_i(x)$$

$$= \sum_{i=1}^{n} c_i L_i \qquad \text{(see A6-2)}.$$

Proof. We use the First Principle of Mathematical Induction (Appendix 3-1), and take for A_n the assertion

$$\lim_{x \to a} \sum_{i=1}^{n} c_i f_i(x) = \sum_{i=1}^{n} c_i L_i .$$

For $n = 1$ we have as assertion A_1,

$$\lim_{x \to a} c_1 f_1(x) = c_1 L_1 ,$$

which is true by Theorem A6-4b.

We now assume A_n true for $n = k$ and seek to prove that A_{k+1} is true. From the induction hypothesis,

$$\lim_{x \to a} \sum_{i=1}^{k} c_i f_i(x) = \sum_{i=1}^{k} c_i L_i .$$

Now

$$\begin{aligned}
\lim_{x \to a} \sum_{i=1}^{k+1} c_i f_i(x) &= \lim_{x \to a}\left(\sum_{i=1}^{k} c_i f_i(x) + c_{k+1}\, f_{k+1}(x)\right) \\
&= \lim_{x \to a} \sum_{i=1}^{k} c_i f_i(x) + \lim_{x \to a} c_{k+1}\, f_{k+1}(x) \quad \text{(Theorem A6-4c)} \\
&= \sum_{i=1}^{k} c_i L_i + \lim_{x \to a} c_{k+1}\, f_{k+1}(x) \\
&= \sum_{i=1}^{k} c_i L_i + c_{k+1}\, L_{k+1} \qquad \text{(Theorem A6-4b)} \\
&= \sum_{i=1}^{k+1} c_i L_i ,
\end{aligned}$$

and assertion A_{k+1} is true. Therefore,

$$\lim_{x \to a} \sum_{i=1}^{n} c_i f_i(x) = \sum_{i=1}^{n} c_i L_i$$

holds for every natural number n.

2. Prove the corollary to Theorem A6-4d. **For any polynomial function p,**

$$\lim_{x \to a} p(x) = p(a)$$

Proof. To establish the corollary, we use the First Principle of Mathematical Induction to prove first that

$$\lim_{x \to a} x^n = a^n, \qquad n \text{ any natural number.}$$

We take for A_n the assertion that $\lim_{x \to a} x^n = a^n$.

For $n = 1$, assertion A_1 is

$$\lim_{x \to a} x = a$$

which is true (Exercises A6-3, No. 2(b)).

We now assume A_k true and try to prove A_{k+1} true. The induction hypothesis is

$$\lim_{x \to a} x^k = a^k.$$

Now

$$\begin{aligned}
\lim_{x \to a} x^{k+1} &= \lim_{x \to a} (x^k \cdot x) \\
&= \left(\lim_{x \to a} x^k\right)\left(\lim_{x \to a} x\right) && \textbf{(Theorem A6-4d)} \\
&= a^k \cdot a && (A_1 \text{ and induction hypothesis}) \\
&= a^{k+1}.
\end{aligned}$$

Thus, A_{k+1} is true and

$$\lim_{x \to a} x^n = a^n, \quad n \text{ any natural number.}$$

To complete the proof, let

$$p(x) = c_0 + c_1x + c_2x^2 + \dots + c_nx^n = \sum_{i=0}^{n} c_ix^i.$$

$$\begin{aligned}
\lim_{x \to a} p(x) &= \lim_{x \to a} \sum_{i=0}^{n} c_ix^i \\
&= \sum_{i=0}^{n} \lim_{x \to a} c_ix^i && \textbf{(Corollary to Theorem A6-4c)} \\
&= \sum_{i=0}^{n} c_i \lim_{x \to a} x^i && \textbf{(Theorem A6-4d)} \\
&= \sum_{i=0}^{n} c_ia^i \\
&= p(a),
\end{aligned}$$

as was to be shown.

3. Prove the corollaries to Lemma A6-4.

(a) Corollary 1. If $\lim_{x \sim a} g(x) = M$ and $M \neq 0$, then there exists a neighborhood of a where $|\frac{3M}{2}| > |g(x)| > |\frac{M}{2}|$ for x in the domain of g.

Proof. Since g has limit M at a, there is a δ-neighborhood of a in which $g(x)$ is closer to M than to zero. We consider two cases: $M > 0$ and $M < 0$.

Case 1. If $M > 0$, we have $g(x) > 0$ by Lemma A6-4, and $\frac{3M}{2} > g(x) > \frac{M}{2} > 0$ or $|\frac{3M}{2}| > |g(x)| > |\frac{M}{2}| > 0$.

Case 2. If $M < 0$, we have $-M > 0$ and $\lim_{x \sim a} \left(-g(x)\right) = -M > 0$. Thus, $-g(x) > 0$ by Lemma A6-4, and $\frac{-3M}{2} > -g(x) > \frac{-M}{2} > 0$. This inequality is equivalent to $|\frac{-3M}{2}| > |-g(x)| > |\frac{-M}{2}| > 0$ or $|\frac{3M}{2}| > |g(x)| > |\frac{M}{2}| > 0$,, which is the same as the inequality obtained in Case 1.

(b) Corollary 2. A limit of a function whose values are nonnegative is nonnegative.

Proof. Let $g(x) \geq 0$ and let $\lim_{x \sim a} g(x) = M$. We want to show that $M \geq 0$. To do this we show that the assumption $M < 0$ leads to a contradiction.

If $M < 0$, then $-M > 0$ and

$$\lim_{x \sim a} \left(-g(x)\right) = -M > 0.$$

Therefore,

$$-g(x) > 0 \qquad \text{(Lemma A6-4)}$$

or

$$g(x) < 0$$

contradicting the hypothesis that $g(x) \geq 0$. Thus, $M \geq 0$.

4. Prove the corollaries to Theorem A6-4e

(a) Corollary 1. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ where $M \neq 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$.

Proof. Since $M \neq 0$, by Theorem 3-4e, we have

$$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M} .$$

From Theorem A6-4d, we have

$$\begin{aligned} \lim_{x \to a} \frac{f(x)}{g(x)} &= \lim_{x \to a} \left(f(x) \cdot \frac{1}{g(x)}\right) \\ &= \left(\lim_{x \to a} f(x)\right) \cdot \left(\lim_{x \to a} \frac{1}{g(x)}\right) \\ &= L \cdot \frac{1}{M} \\ &= \frac{L}{M} . \end{aligned}$$

(b) Corollary 2. If p and q are polynomials, and if $q(a) \neq 0$, then

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} .$$

Proof. By the corollary to Theorem A6-4d,

$$\lim_{x \to a} p(x) = p(a) \quad \text{and} \quad \lim_{x \to a} q(x) = q(a).$$

Since $q(a) \neq 0$,

$$\lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)} \text{ (Corollary 1 to Theorem A6-4e).}$$

5. Find the following limits, giving at each step the theorem on limits which justifies it.

(a) $\lim_{x \to 3} (2 + x) = \lim_{x \to 3} 2 + \lim_{x \to 3} x$ Theorem A6-4c

$= 2 + 3$ Theorem A6-4a
Example A6-3a

$= 5.$

(b) $$\lim_{x \to -1} (5x - 2) = \lim_{x \to -1} 5x + \lim_{x \to -1} (-2)$$ Theorem A6-4c

$$= -5 + (-2)$$ Theorem A6-4b
Example A6-3a
Theorem A6-4a

$$= -7.$$

(c) $\lim_{x \to 0} \left(\frac{a}{1 + |x|} - b\sqrt{|x|}\right)$, where a and b are constants.

$$\lim_{x \to 0} \left(\frac{a}{1 + |x|} - b\sqrt{|x|}\right) = \lim_{x \to 0} \frac{a}{1 + |x|} + \lim_{x \to 0} (-b\sqrt{|x|})$$ Theorem A6-4c

$$= a \lim_{x \to 0} \frac{1}{1 + |x|} - b \lim_{x \to 0} \sqrt{|x|}$$ Theorem A6-4b

$$= a \cdot 1 - b \cdot 0$$ Example A6-3b
Theorem A6-4e
Example A6-3c

$$= a.$$

(d) $\lim_{x \to a} (x^3 + ax^2 + a^2x + a^3)$, where a is constant.

$$\lim_{x \to a} (x^3 + ax^2 + a^2x + a^3) = a^3 + a \cdot a^2 + a^2 \cdot a + a^3$$ Corollary to
Theorem A6-4d

$$= 4a^3.$$

6. Find the following limits, giving at each step the theorem which justifies it.

(a) $$\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{(x - 1)(x + 1)}$$

$$= \lim_{x \to 1} \left(\frac{x - 1}{x - 1}\right) \cdot \lim_{x \to 1} \left(\frac{x^2 + x + 1}{x + 1}\right)$$ Theorem A6-4d

$$= 1 \cdot \frac{3}{2}$$ Corollary 2 to
Theorem A6-4e
Corollary to
Theorem A6-4d

$$= \frac{3}{2}.$$

(b) $$\lim_{x\to 3} \frac{x^2 - 9}{x^3 - 27} = \lim_{x\to 3} \frac{(x - 3)(x + 3)}{(x - 3)(x^2 + 3x + 9)}$$

$$= \lim_{x\to 3} \left(\frac{x - 3}{x - 3}\right) \cdot \lim_{x\to 3}\left(\frac{x + 3}{x^2 + 3x + 9}\right)$$

Theorem A6-4d

$$= 1 \cdot \frac{6}{27}$$

Corollary 2 to
Theorem A6-4e
Corollary to
Theorem A6-4d

$$= \frac{2}{9}.$$

7. Find $\lim_{x\to 1} \frac{x^n - 1}{x - 1}$, for n a positive integer. Verify first that

$$\frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \dots + x + 1 \quad (x \neq 1).$$

The verification of the division is done by simple algebraic methods.

Now let $p(x) = x^{n-1} + x^{n-2} + \dots + x + 1$.

$$\lim_{x\to 1} p(x) = p(1)$$

Corollary to
Theorem A6-4d

$$= n.$$

8. Determine whether the following limits exist and, if they do exist, find their values.

(a) $\lim_{x\to 1} \frac{1 + \sqrt{x}}{1 - x}$ does not exist.

$\lim_{x\to 1} \frac{1 + \sqrt{x}}{1 - x} = \lim_{x\to 1} \frac{1}{1 - \sqrt{x}}$ which does not exist.

(b) $\lim_{x\to a} (x^n - a^n)$, n a positive integer, a constant.

Let $p(x) = x^n - a^n$.

$$\lim_{x\to a} p(x) = p(a)$$

$$= a - a$$

$$= 0.$$

(c) $\lim\limits_{x \to -1} \dfrac{\sqrt{2 + x} + 1}{x + 1}$

$\lim\limits_{x \to -1} (\sqrt{2 + x} + 1) = 2$ and $\lim\limits_{x \to -1} (x + 1) = 0$, hence $\lim\limits_{x \to -1} \dfrac{\sqrt{2 + x} + 1}{x + 1}$ does not exist.

(d) $$\lim_{x \to 1} \frac{(x - 2)(\sqrt{x} - 1)}{x^2 + x - 2} = \lim_{x \to 1} \frac{(x - 2)(\sqrt{x} - 1)}{(x + 2)(x - 1)}$$

$$= \lim_{x \to 1} \frac{(x - 2)(\sqrt{x} - 1)}{(x + 2)(\sqrt{x} + 1)(\sqrt{x} - 1)}$$

$$= \lim_{x \to 1} \frac{x - 2}{(x + 2)(\sqrt{x} + 1)} = -\frac{1}{6}.$$

(e) $\lim\limits_{x \to 1} \dfrac{1 - \sqrt{x}}{1 - x} = \lim\limits_{x \to 1} \dfrac{1}{1 + \sqrt{x}} = \dfrac{1}{2}$.

9. Using the algebra of limits show that $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} = L$ if and only if $\lim\limits_{x \to a} \dfrac{f(x) - f(a) - L(x - a)}{|x - a|} = 0$.

First, we note that $\lim\limits_{x \to a} g(x) = 0$ if and only if $\lim\limits_{x \to a} |g(x)| = 0$, and also that $|g(x)| = \big||g(x)|\big|$.

Let

and
$$A = \frac{f(x) - f(a)}{x - a} - L,$$

$$B = \frac{f(x) - f(a) - L(x - a)}{|x - a|}.$$

Then,

$$|A| = |B|.$$

Part 1. Assume $\lim\limits_{x \to a} B = 0$. This implies that $\lim\limits_{x \to a} |A| = 0$ and, thus,

$$\lim_{x \to a} A = 0 \quad \text{or} \quad \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = L.$$

Part 2. Assume $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} = L$. This implies that $\lim\limits_{x \to a} A = 0$ and, thus

$$\lim_{x \to a} |B| = 0 \quad \text{and} \quad \lim_{x \to a} B = 0.$$

10. Assume $\lim\limits_{x \sim 0} \sin x = 0$ and $\lim\limits_{x \sim 0} \cos x = 1$. **Find each of the following** limits, if the limit exists, giving at each step the theorem on limits which justifies it.

(a)
$$\begin{aligned}\lim_{x \sim 0} \sin^3 x &= \left(\lim_{x \sim 0} \sin^2 x\right) \cdot \left(\lim_{x \sim 0} \sin x\right) && \text{Theorem A6-4d}\\ &= \left(\lim_{x \sim 0} \sin x \cdot \lim_{x \sim 0} \sin x\right) \cdot \left(\lim_{x \sim 0} \sin x\right) && \text{Theorem A6-4d}\\ &= 0.\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x \sim 0} \tan x &= \lim_{x \sim 0} \frac{\sin x}{\cos x}\\ &= \frac{\lim\limits_{x \sim 0} \sin x}{\lim\limits_{x \sim 0} \cos x} = 0. && \text{Corollary 1 to Theorem A6-4e}\end{aligned}$$

(c)
$$\begin{aligned}\lim_{x \sim 0} \sin 2x &= \lim_{x \sim 0} (2 \sin x \cos x)\\ &= 2 \cdot \left(\lim_{x \sim 0} \sin x \cos x\right) && \text{Theorem A6-4b}\\ &= 2 \cdot \left(\lim_{x \sim 0} \sin x\right) \cdot \left(\lim_{x \sim 0} \cos x\right) && \text{Theorem A6-4d}\\ &= 0.\end{aligned}$$

(d)
$$\begin{aligned}\lim_{x \sim 0} \frac{\sin x}{\tan x} &= \lim_{x \sim 0} \frac{\sin x \cdot \cos x}{\sin x}\\ &= \lim_{x \sim 0} \frac{\sin x}{\sin x} \cdot \lim_{x \sim 0} \cos x && \text{Theorem A6-4d}\\ &= 1.\end{aligned}$$

(e)
$$\begin{aligned}\lim_{x \sim 0} \frac{1 - \cos x}{\sin x} &= \lim_{x \sim 0} \frac{1 - \cos x}{\sin x} \cdot \frac{1 + \cos x}{1 + \cos x} && \text{Theorem A6-4d}\\ &= \lim_{x \sim 0} \frac{\sin^2 x}{\sin x} \cdot \lim_{x \sim 0} \frac{1}{1 + \cos x} && \text{Theorem A6-4d}\\ &= 0 \cdot \frac{1}{2} = 0.\end{aligned}$$

(f) $$\lim_{x\sim 0} \frac{\cos 2x}{\cos x + \sin x} = \lim_{x\sim 0} \frac{\cos^2 x - \sin^2 x}{\cos x + \sin x}$$

$$= \lim_{x\sim 0} (\cos x - \sin x) \cdot \lim_{x\sim 0} \frac{\cos x + \sin x}{\cos x + \sin x}$$

Theorem A6-4d

$$= \left[\lim_{x\sim 0} \cos x - \lim_{x\sim 0} \sin x\right] \cdot 1$$

Theorem A6-4c

$$= 1.$$

11. Prove the corollaries to Theorem A6-4f.

(a) Corollary 1 (Sandwich Theorem). If $h(x) \leq f(x) \leq g(x)$ in some deleted neighborhood of a, and if $\lim_{x\sim a} h(x) = K$ and $\lim_{x\sim a} g(x) = M$, then, if $\lim_{x\sim a} f(x)$ exists, $K \leq \lim_{x\sim a} f(x) \leq M$.

Pooof. Since $f(x) \leq g(x)$ in a deleted neighborhood of a, we have, by Theorem A6-4f, that

$$\lim_{x\sim a} f(x) \leq \lim_{x\sim a} g(x).$$

Again, since $h(x) \leq f(x)$ in a deleted neighborhood of a, we have, by Theorem A6-4f, that

$$\lim_{x\sim a} h(x) \leq \lim_{x\sim a} f(x).$$

Thus,

$$\lim_{x\sim a} h(x) \leq \lim_{x\sim a} f(x) \leq \lim_{x\sim a} g(x)$$

or

$$K \leq \lim_{x\sim a} f(x) \leq M.$$

(b) Corollary 2 (Squeeze Theorem). (Hint: Prove $\lim_{x\sim a} f(x)$ exists.)

Because the proof of the Squeeze Theorem does not follow immediately from Theorem A6-4f it is given in Section TCA6-4 immediately preceding Solutions Exercises A6-4.

12. For what integral values of m and n does $\lim\limits_{x\sim -a} \dfrac{x^m + a^m}{x^n + a^n}$ exist? Find the limit for these cases.

Part 1. For $a = 0$:

(i) if $m > n$,

$$\lim_{x\sim 0} \frac{x^m}{x^n} = \lim_{x\sim 0} x^{m-n} = 0;$$

(ii) if $m = n$,

$$\lim_{x\sim 0} \frac{x^m}{x^n} = \lim_{x\sim 0} 1 = 1;$$

(iii) if $m < n$, $\lim\limits_{x\sim 0} \dfrac{x^m}{x^n}$ does not exist.

Part 2. For $a \neq 0$:

$$\lim_{x\sim -a} (x^p + a^p) = \begin{cases} 2a^p , & p \text{ even} \\ 0 , & p \text{ odd.} \end{cases}$$

(i) Thus, for $a \neq 0$ and n even,

$$\lim_{x\sim -a} \frac{x^m + a^m}{x^n + a^n} = \frac{\lim\limits_{x\sim -a} (x^m + a^m)}{\lim\limits_{x\sim -a} (x^n + a^n)}$$

$$= \begin{cases} \dfrac{2a^m}{2a^n} = a^{m-n} , & m \text{ even} \\ \dfrac{0}{2a^n} = 0 , & m \text{ odd.} \end{cases}$$

(ii) For $a \neq 0$, n odd, m even, $\lim\limits_{x\sim -a} \dfrac{x^m + a^m}{x^n + a^n}$ does not exist.

(iii) For $a \neq 0$, n odd, m odd,

$$\lim_{x \to -a} \frac{x^m + a^m}{x^n + a^n} = \lim_{x \to -a} \frac{(x + a)(x^{m-1} - ax^{m-2} + \ldots + a^{m-1})}{(x + a)(x^{n-1} - ax^{n-2} + \ldots + a^{n-1})}$$

$$= \frac{ma^{m-1}}{na^{n-1}}$$

$$= \frac{m}{n} a^{m-n}$$

13. Prove that if $\lim\limits_{x \to a} f(x) = 0$ and $g(x)$ is bounded in a neighborhood of $x = a$, then $\lim\limits_{x \to a} f(x) \cdot g(x) = 0$.

Proof. Since $g(x)$ is bounded there exists a positive number M such that

$$-M < g(x) < M$$

in a neighborhood of $x = a$. Consequently,

$$-M \cdot |f(x)| \leq f(x)g(x) \leq M \cdot |f(x)|$$

in a neighborhood of $x = a$.

By the Squeeze Theorem,

$$\lim_{x \to a} f(x) \cdot g(x) = 0.$$

14. (a) Verify that if $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ exists and if $\lim\limits_{x \to a} g(x) = 0$, then $\lim\limits_{x \to a} f(x) = 0$.

$$\lim_{x \to a} \frac{f(x)}{g(x)} \cdot \lim_{x \to a} g(x) = \lim_{x \to a} \left(\frac{f(x)}{g(x)} \cdot g(x) \right)$$

or

$$0 = \lim_{x \to a} f(x). \qquad \text{Theorem A6-4d}$$

(b) Give examples of functions f and g for which $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ yet the limit of their quotient does not exist.

If $f(x) = x^m$ and $g(x) = x^n$, $m < n$, $\frac{f(x)}{g(x)}$ does not have a limit at 0. Of course, many other examples exist.

15. Prove that if $\lim_{x \to a} g(x) = 0$ and $\lim_{x \to a} f(x)$ does not exist, then the limit of the quotient $\frac{f(x)}{g(x)}$ does not exist.

If $\lim_{x \to a} \frac{f(x)}{g(x)}$ does exist and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} f(x) = 0$ by by Number 14(a). Contradiction.

16. The right-hand limit at a point $P(p, f(p))$ of a function is the limit of the function at the point P for a right-hand domain $(p, p + \delta)$. Similarly, for the left-hand limit, the domain is restricted to $(p - \delta, p)$. We denote them symbolically by $\lim_{x \to p^+} f(x)$ and $\lim_{x \to p^-} f(x)$, respectively.

In particular, $\lim_{x \to 2^+} [x] = 2$, $\lim_{x \to 2^-} [x] = 1$. Determine the indicated limits, if they exist, of the following:

(a) $\lim_{x \to 2^+} \frac{[x]^2 - 4}{x^2 - 4}$.

For $x \in (2, 2 + \delta)$, $0 < \delta < 1$,

$$[x]^2 - 4 = ([x] - 2)([x] + 2)$$
$$= 0 \cdot 4 = 0.$$

Thus

$$\lim_{x \to 2^+} \frac{[x]^2 - 4}{x^2 - 4} = \lim_{x \to 2^+} 0 = 0.$$

(b) $\lim_{x \to 2^-} \frac{[x]^2 - 4}{x^2 - 4}$.

For $x \in (2 - \delta, 2)$, $0 < \delta < 1$,

$$[x]^2 - 4 = (-1)(3) = -3.$$

Thus

$$\lim_{x \to 2^-} \frac{[x]^2 - 4}{x^2 - 4} = \lim_{x \to 2^-} \frac{3}{x^2 - 4}$$

does not exist.

(c) $\lim\limits_{x \sim 3^+} (x - 2 + [2 - x] - [x])$.

For $x \in (3, 3 + \delta)$, $0 < \delta < 1$,

$$[2 - x] = -2 \quad \text{and} \quad [x] = 3.$$

Thus

$$\lim_{x \sim 3^+} (x - 2 + [2 - x] - [x]) = \lim_{x \sim 3^+} (x - 2 - 2 - 3)$$
$$= -4.$$

(d) $\lim\limits_{x \sim 3^-} (x - 2 + [2 - x] - [x])$.

For $x \in (3 - \delta, 3)$, $0 < \delta < 1$,

$$[2 - x] = -1 \quad \text{and} \quad [x] = 2.$$

Thus,

$$\lim_{x \sim 3^-} (x - 2 + [2 - x] - [x]) = \lim_{x \sim 3^-} (x - 2 - 1 - 2)$$
$$= -2.$$

(e) $\lim\limits_{x \sim 0^+} \left(\frac{x}{a}\left[\frac{b}{x}\right] - \frac{b}{x}\left[\frac{x}{a}\right]\right)$, $a > 0$, $b > 0$.

Since $b > 0$, we can write $\frac{b}{x} = n + r$, where $n \le \frac{b}{x}$ is a non-negative integer and $0 \le r < 1$. Thus $\left[\frac{b}{x}\right] = n$, $x = \frac{b}{n + r}$, whence

$$\frac{x}{a}\left[\frac{b}{x}\right] = \frac{bn}{a(n + r)}$$
$$= \frac{b}{a(1 + \frac{r}{n})}.$$

As $x \sim 0^+$, n increases without bound and $\frac{r}{n} \sim 0$. Since $a > 0$, $\left[\frac{x}{a}\right] = 0$ for $0 < x < a$. Thus,

$$\lim_{x \sim 0^+} \left(\frac{x}{a}\left[\frac{b}{x}\right] - \frac{b}{x}\left[\frac{x}{a}\right]\right) = \lim_{x \sim 0^+} \frac{x}{a}\left[\frac{b}{x}\right] - \lim_{x \sim 0^+} \frac{b}{x}\left[\frac{x}{a}\right]$$
$$= \frac{b}{a} - \lim_{x \sim 0^+} 0$$
$$= \frac{b}{a}.$$

(f) $\lim_{x \sim 0^-} \left(\frac{x}{a} \left[\frac{b}{x}\right] - \frac{b}{x} \left[\frac{x}{a}\right] \right)$, $a > 0$, $b > 0$.

The first term is similar to the first term in (e) except that n is a negative integer. Since $a > 0$, $\left[\frac{x}{a}\right] = -1$ for $0 < |x| < a$; however, $\left|\frac{b}{x}\right|$ increases without bound as $x \sim 0^-$. Thus $\lim_{x \sim 0^-} \left(\frac{x}{a} \left[\frac{b}{x}\right] - \frac{b}{x} \left[\frac{x}{a}\right] \right)$ does not exist.

(g) $\lim_{x \sim 0^+} \frac{\sqrt{x}}{\sqrt{4 + \sqrt{x}} - 2}$.

$$\lim_{x \sim 0^+} \frac{\sqrt{x}}{\sqrt{4 + \sqrt{x}} - 2} = \lim_{x \sim 0^+} \frac{\sqrt{x}}{\sqrt{4 + \sqrt{x}} - 2} \cdot \frac{\sqrt{4 + \sqrt{x}} + 2}{\sqrt{4 + \sqrt{x}} + 2}$$

$$= \lim_{x \sim 0^+} \frac{\sqrt{x}}{\sqrt{x}} \cdot \left(\sqrt{4 + \sqrt{x}} + 2\right)$$

$$= 4.$$

Appendix 7

CONTINUITY THEOREM

TC A7-1. Completeness of the Real Number System. The Separation Axiom

The completeness of the real number system is a consequence of the Separation Axiom. We note that in the axiom the sets A and B might not be disjoint; e.g.,

$$A = \{x : x \leq 0\} , B = \{x : x \geq 0\}$$

and 0 is the unique separation number. Observe that the word "separates" as used in the separation axiom does not mean that s is not in either A or B. The separation number s may be in either set, both sets, or neither set.

The completeness of the field of real numbers plays a central role in the rigorous development of the calculus. We have based our logical development upon the Separation Axiom because of its intuitive geometric content. It asserts the absence of gaps (holes) in the real number line. The Least Upper Bound Principle and the Separation Axiom are logically equivalent. Thus, in the sequel, we can feel free to use whichever is most convenient.

We have approached the real numbers axiomatically; i.e., we have given a set of axioms rich enough to yield the properties that we need. An alternative approach is to define the set of real numbers in terms of a more basic set, say the rational numbers. This approach was taken in the nineteenth century by Dedekind, Cantor, and others. The original article of Dedekind now appears in the paperback, Essays in the Theory of Numbers, by R. Dedekind, Dover, 1963. His account is both lucid and elementary.

Solutions Exercises A7-1

1. Prove Corollary 1 to the Least Upper Bound Principle. If M is the least upper bound of the set A, then for each positive ϵ there exists an $\alpha \in A$ such that $\alpha > M - \epsilon$.

   Suppose there is no $\alpha > M - \epsilon$. Then $M' : M - \epsilon < M' < M$ is an upper bound for A, contradicting M as the least upper bound.

2. Prove Corollary 2 to the Least Upper Bound Principle. A set of numbers which is bounded below has a greatest lower bound.

   For the proof of Corollary 2, let $\overline{A}$ be a set which is bounded below and let $\overline{B}$ be the set of lower bounds of $\overline{A}$. The set $A = \{-\alpha : \alpha \in \overline{A}\}$ has as the set of its upper bounds, $B = \{-\beta : \beta \in \overline{B}\}$. The greatest lower bound $\overline{M}$ of $\overline{A}$ is given by $\overline{M} = -M$ where M is the least upper bound of A.

3. (a) Consider the sets A of positive rational numbers α satisfying $\alpha^2 < 2$,, and B of positive rational numbers β satisfying $\beta^2 > 2$. Prove if $\alpha \in A$ and $\beta \in B$ that $\alpha < \beta$.

   If $\alpha = \beta$, then $\alpha^2 = \beta^2$ contradicting $\alpha^2 < \beta^2$.
   If $\alpha > \beta$, then since $\alpha > 0$ and $\beta > 0$, we have $\alpha^2 > \beta^2$.
   These contradictions force the conclusion $\alpha < \beta$.

   (b) Show that a separation number s for the sets A and B must satisfy $s^2 = 2$; i.e., $s = \sqrt{2}$.

   Suppose s is a separation number for A and B and $s \neq \sqrt{2}$, say $s = \sqrt{2} + \epsilon$, $\epsilon > 0$. Then, $\sqrt{2} + \frac{\epsilon}{2} < s$ but $\sqrt{2} + \frac{\epsilon}{2} \in B$, since $(\sqrt{2} + \frac{\epsilon}{2})^2 = 2 + \epsilon\sqrt{2} + \frac{\epsilon^2}{4} > 2$. This contradiction indicates $s = \sqrt{2} + \epsilon$ is not a separation number. $s = \sqrt{2} - \epsilon$ can also be shown to fail, leaving only $s = \sqrt{2}$ as the separation number.

(c) Prove that $\sqrt{2}$ is irrational.

Assume $\frac{p}{q}$ is a rational solution in lowest terms of $x^2 = 2$. Then, $p^2 = 2q^2$ implies p^2 is even and thus $p = 2r$, where r is an integer. Now $4r^2 = 2q^2$ implies q is even which contradicts our original assumption that $\frac{p}{q}$ is in lowest terms. Hence $\sqrt{2}$ is irrational.

4. (a) Prove for every real number a, that there is an integer n greater than a (Principle of Archimedes).

Recall that by definition each integer n has a succeeding integer $n + 1$. Suppose there were no integer greater than a. Then, since the integers would have an upper bound a, they would have a least upper bound M. The number $M - \frac{1}{2}$ would not be an upper bound since M is least. Consequently there is an integer $n > M - \frac{1}{2}$. Then $n + 1 > M + \frac{1}{2} > M$, contrary to the definition of least upper bound.

(b) Prove that given any $\epsilon > 0$ there is an integer n such that $0 < \frac{1}{n} < \epsilon$.

Part (a) proved that there is an integer $n > \frac{1}{\epsilon}$. Hence $\epsilon > \frac{1}{n} > 0$.

5. (a) We define the infinite decimal

$$c_0 . c_1 c_2 c_3 \dots ,$$

where c_0 is an integer, and $c_1, c_2, c_3, \dots$, are digits as the number r where

$$c_0 + \frac{c_1}{10} + \frac{c_2}{10^2} + \dots + \frac{c_n}{10^n} \leq r < c_0 + \frac{c_1}{10} + \frac{c_2}{10^2} + \dots + \frac{c_n + 1}{10^n} .$$

Show that the preceding inequality does, in fact, define a unique real number.

Define two sets, A_1 the set of all a_n, and B_1 the set of all b_n, so that

$$a_n = c_0 + \frac{c_1}{10} + \dots + \frac{c_n}{10^n}$$

$$b_n = c_0 + \frac{c_1}{10} + \dots + \frac{c_n + 1}{10^n} .$$

We have $a_n \leq r < b_n$ for all n. Then r is a separation number for A, and B. Given any $\epsilon > 0$, then $b_n - a_n = \frac{1}{10^n} < \epsilon$ for n sufficiently large. By Lemma A7-1, r is the unique separation number for A and B.

(b) Given a real number r we define its decimal representation recursively in terms of the integer part function $[x]$ as follows:

$$c_0 = [r]$$

$$c_n = \left[10^n\left(r - c_0 - \frac{c_1}{10} - \frac{c_2}{10^2} - \dots - \frac{c_{n-1}}{10^{n-1}}\right)\right].$$

Show that the inequality in part (a) is satisfied for this choice of c_n. Show also that decimals consisting entirely of 9's from some point on are avoided. (Thus, we obtain $2 = 2.000 \dots$ but not $2 = 1.999 \dots$)

Since $x - 1 < [x] \leq x$,

$$(1) \quad 10^n\left(r - c_0 - \dots - \frac{c_{n-1}}{10^{n-1}}\right) - 1 < c_n$$

$$\leq 10^n\left(r - c_0 - \dots - \frac{c_{n-1}}{10^{n-1}}\right),$$

or equivalently

$$(2) \quad c_0 + \dots + \frac{c_n}{10^n} \leq r < c_0 + \dots + \frac{c_n + 1}{10^n} .$$

(Note the strong inequality on the right.)

Next, we establish for $n \geq 1$ that c_n is a digit by mathematical indication.

For c_1, we have from (1) that $10(r - c_0) - 1 < c_1 \leq 10(r - c_0)$ where $0 \leq (r - c_0) < 1$.

Hence, $-1 < c_1 < 10$

or, since c_1 is an integer ≥ 0, $0 \leq c_1 \leq 9$. Now suppose that c_n is a digit $0 \leq c_n \leq 9$. In the notation of Number 5a, we see from (1) that

$$(3) \qquad c_n \leq 10^n(r - a_{n-1}) < 1 + c_n .$$

Now, replacing n by $n + 1$ in (1) we have

$$(4) \quad 10^{n+1}\left(r - a_{n-1} - \frac{c_n}{10^n}\right) - 1 < c_{n+1} \leq 10^{n+1}\left(r - a_{n-1} - \frac{c_n}{10^n}\right).$$

Using the inequality (3) in (4) we obtain

$$-1 < c_{n+1} < 10$$

from which we conclude that c_{n+1} is a digit.

Now, let us suppose that r can be represented as a decimal with an infinite string of 9's

$$r = d_0 . d_1 d_2 \cdots d_{p-1}\, d_p\, 999 \cdots$$

where we may without loss of generality suppose that either $p = 0$ or $d_p \neq 9$ (i.e., that the last decimal place where a 9 does not appear, if there is any, is the p-th place). Consider the number

$$\rho = d_0 . d_1 d_2 \cdots (1 + d_p)\, 000 \cdots .$$

Take $d_q = 9$ for $q > p$. For any index q, then, we see that both ρ and r lie between the numbers

$$d_0 + \frac{d_1}{10} + \cdots + \frac{d_q}{10^q}$$

and
$$d_0 + \frac{d_1}{10} + \cdots + \frac{1 + d_q}{10^q} .$$

Since the upper and lower estimates here differ by $\frac{1}{10^q}$ it follows that $\rho = r$. Now the method of representation given above yields a unique decimal, since (2) is equivalent to the definition of d_n by means of the integer part function. Taking $c_q = d_q$ for $q < p$, $c_p = 1 + d_p$, and $c_q = 0$ for $q > p$ we see that equality holds on the left in (2) for $q \geq p$; hence any other representation then the terminating one is precluded.

6. An infinite decimal $c_0 . c_1c_2c_3 \dots$ is said to be periodic if for some fixed value p, the period of the decimal, we have $c_{n+p} = c_n$ for all n satisfying $n \geq n_0$, where we require that p is the smallest positive integer satisfying this condition. In words, from some place on, the decimal consists of the indefinite repetition of the same p digits. Thus

$$\frac{1}{3} = .33333\cdots$$

$$\frac{15}{44} = .34090909\cdots$$

are periodic decimals. It is convenient to indicate a cycle of p digits by underlining, rather than repetition; e.g.,

$$\frac{22}{7} = 3.\underline{142857} .$$

(a) **Prove that every periodic decimal represents a rational number.** (Hint: Consider the decimal as a geometric progression.)

Let $r = c_0 . c_1 \dots . c_q \underline{b_1b_2 \dots b_p}$

Then $r = \dfrac{\gamma}{10^q} + \dfrac{\beta}{10^{p+q}} + \dfrac{\beta}{10^{2p+q}} + \dfrac{\beta}{10^{3p+q}} + \cdots$

where $\gamma = 10^q\,(c_0 . c_1c_2 \cdots c_q)$.

and $\beta = 10^p\,(0 . b_1b_2 \cdots b_p)$.

This latter representation for r is an infinite geometric series with common ratio $\dfrac{1}{10^p}$. Whence we have

$$r = \frac{\gamma}{10^q} + \frac{\beta}{10^{p+q}\,(1 - 10^{-p})}$$

$$= \frac{(10^p - 1)\gamma + \beta}{10^q(10^p - 1)} ,$$

which is a rational number, since $(10^p - 1)\gamma + \beta$ and $10^q(10^p - 1)$ are integers.

(b) Prove that every rational number has a periodic decimal representation. (A "terminating" decimal in which each place beyond a certain point is zero is considered as a special case of periodic decimals.) If $r = \frac{s}{t}$ represents a rational number given in lowest terms, find the largest possible period of the infinite decimal representation of r in terms of the denominator t.

From (a) and (b) we conclude that a decimal which is not periodic represents an irrational number, and conversely.

We first submit a specific case which contains the germ of the proof: we discuss the rational number $\frac{8}{7}$.

$$\frac{8}{7} = 1 + \frac{1}{7}$$

$$\frac{1}{7} = \frac{1}{10}\left(\frac{10}{7}\right) = \frac{1}{10}\left(1 + \frac{3}{7}\right),$$

$$\frac{3}{7} = \frac{1}{10}\left(\frac{30}{7}\right) = \frac{1}{10}\left(4 + \frac{2}{7}\right),$$

$$\frac{2}{7} = \frac{1}{10}\left(\frac{20}{7}\right) = \frac{1}{10}\left(2 + \frac{6}{7}\right),$$

$$\frac{6}{7} = \frac{1}{10}\left(\frac{60}{7}\right) = \frac{1}{10}\left(8 + \frac{4}{7}\right),$$

$$\frac{4}{7} = \frac{1}{10}\left(\frac{40}{7}\right) = \frac{1}{10}\left(5 + \frac{5}{7}\right),$$

$$\frac{5}{7} = \frac{1}{10}\left(\frac{50}{7}\right) = \frac{1}{10}\left(7 + \frac{1}{7}\right).$$

Compare this procedure with the "long division" process:

$$\begin{array}{r@{}l}
 & .142857 \\
7 & \overline{)1.000000} \\
 & \underline{7} \\
 & 30 \\
 & \underline{28} \\
 & 20 \\
 & \underline{14} \\
 & 60 \\
 & \underline{56} \\
 & 40 \\
 & \underline{35} \\
 & 50 \\
 & \underline{49} \\
 & 1\,.
\end{array}$$

Note that we stop with a remainder 1 since it occured before. We would get a repetition of the same digits if we were to continue the division process:

$$\frac{8}{7} = 1.142857142857\ldots .$$

In general we shall always get a remainder which is repeated since there are only a finite numbers of different possible remainders. The argument given here is based on the division algorithm which must eventually repeat itself because remainders must recur.

Let $r = \frac{\sigma}{\tau}$ where σ and τ are relatively prime and $\tau > 0$. Let the decimal expansion of r as given by the method of Number 5 be

$$c_0 . c_1c_2c_3 \cdots .$$

Represent τ in the form 2^a5^bm where m has no factors of 2 or 5. Set $q = \max\{a,b\}$, $t = 10^qm$ and rewrite r in the form

$$r = \frac{s}{t} = \frac{s}{10^qm} .$$

Observe that m and s are relatively prime.
If $m = 1$ we have exactly

$$r = \frac{s}{10^q} = c_0 . c_1c_2 \cdots c_q$$

so that $c_n = 0$ for $n > q$ and the period is 1. If $m \neq 1$, take

$$\lambda_k = 10^{q+k} (c_0 . c_1c_2 \cdots c_{q+k}).$$

From (2) in the solution of Number 5, we have

$$\lambda_k \leq \frac{10^ks}{m} < \lambda_k + 1,$$

whence

$$0 \leq 10^ks - m\lambda_k < m.$$

Consequently, on dividing 10^ks by m we obtain the quotient λ_k and the remainder $r_k = 10^ks - m\lambda_k$. Now $r_k \neq 0$ since m and 10^ks are relatively prime. Thus r_k, as a remainder on division by m can only be one of the integers 1, 2, ..., $m - 1$. It follows that at least two of the m numbers r_k, for $k = 1, 2, \ldots, m$, must be the same, say $r_i = r_j$ with $j > i$. From this we now prove that the decimal expansion for r is periodic with the period $p = j - i$, namely,

$$r = c_0 . c_1c_2 \cdots c_{q+i} \underline{c_{q+i+1} \cdots c_{q+i+p}} .$$

We show first that $r_k = c_{k+p}$. Observe that,

$$r_{k+1} = 10^{k+1} s - m\lambda_{k+1} = 10^{k+1} s - m(10\lambda_k + c_{q+k+1})$$

$$= 10r_k - m\, c_{q+k+1}.$$

But $$c_{q+k+1} = \left[10^{q+k+1} (r - c_0 . c_1 c_2 \ldots c_{q+k})\right]$$

$$= \left[10(\frac{10^k s}{m} - \lambda_k)\right]$$

$$= \left[\frac{10r_k}{m}\right].$$

Combining the last two results, we obtain a representation for r_{k+1} in terms of r_k alone. It follows from $r_i = r_j$ that $r_{i+1} = r_{j+1}$, $r_{i+2} = r_{j+2}$, etc. Thus r_k is periodic with period p. Since c_{q+k+1} is a function of r_k we see that the decimal also is periodic with period p.

(c) Prove for every positive prime α other than 2 and 5 that there exists an integer, all of whose digits are ones, for which α is a factor; i.e., α is a factor of some number of the form

$$10^n + 10^{n-1} + 10^{n-2} + \ldots + 10 + 1.$$

Let α be the given prime. We can write (from part (a))

$$\frac{1}{\alpha} = \frac{(10^p - 1)\gamma + \beta}{10^q(10^p - 1)}$$

or

$$\alpha\Big((10^p - 1)\gamma + \beta\Big) = 10^q(10^p - 1).$$

Since α is neither 2 nor 5 it follows that α is a factor of

$$10^p - 1 = 9(10^{p-1} + 10^{p-2} + \ldots + 10 + 1).$$

If $\alpha \neq 3$, then α must be a factor of the expression in parenthesis. If $\alpha = 3$, then α is a factor of $10^2 + 10 + 1$. In either case, the result is proved.

7. (a) Consider a polynomial with integer coefficients:

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 . \qquad (a_n \neq 0)$$

Prove that if $\frac{p}{q}$ is a rational root of this polynomial given in lowest terms, then p is a factor of a_0 and q is a factor of a_n.

If $\frac{p}{q}$ is a root of the polynomial, then

$$a_n \frac{p^n}{q^n} + a_{n-1} \frac{p^{n-1}}{q^{n-1}} + \dots + a_1 \frac{p}{q} + a_0 = 0,$$

whence, on multiplying by q^n,

$$a_n p^n + a_{n-1} p^{n-1} q + \dots + a_1 p q^{n-1} + a_0 q^n = 0.$$

It follows that p is a factor of $a_0 q^n$ and q is a factor of $a_n p^n$. Since p and q have no factors in common, we conclude that p is a factor of a_0 and q a factor of a_n.

(b) Show that $x^3 + x + 1$ has no rational root.

By the preceding result the only conceivable rational roots are 1 and -1 and neither is a root.

(c) Prove that if $\sqrt{n}$ is rational then it is integral.

A rational root $\frac{p}{q}$ of $x^2 - n = 0$ must be an integer divisor of n, since $q = \pm 1$.

(d) Prove that $\sqrt{3} - \sqrt{2}$ is irrational.

Set $a = \sqrt{3} - \sqrt{2}$. Squaring we obtain

$$a^2 = 5 - 2\sqrt{6}$$

whence

$$(a^2 - 5) = -2\sqrt{6}$$

and

$$(a^2 - 5)^2 = 24$$

or

$$a^4 - 10a^2 + 1 = 0.$$

The only conceivable rational roots of this equation are ± 1 and neither is a root.

Alternatively. Assume $\sqrt{3} - \sqrt{2} = r$ (rational); then,

$$r^2 - 2r\sqrt{2} + 2 = 3.$$

Since this implies that $\sqrt{2}$ is rational, $\sqrt{3} - \sqrt{2}$ is irrational.

Solutions Exercises A7-2

1. Let f(x) $$f(x) = \begin{cases} \sin \frac{1}{x}, & x > 0 \\ 0, & x = 0 \end{cases}$$

Show that f satisfies the conclusion of Theroem 8-2a on any interval $[0,b]$ but f is not continuous at $x = 0$.

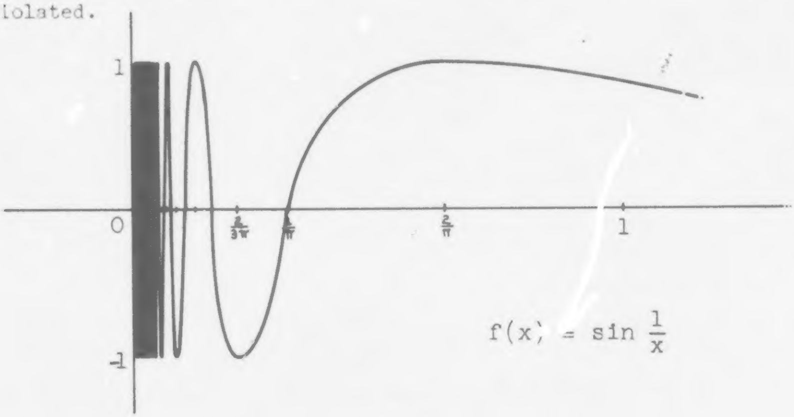
$g(y) = \sin y$ is a periodic function of period 2π, so it will take on all of the values $-1 \leq g(y) \leq 1$ on any interval $n\pi \leq y \leq (n + 2)\pi$. Certainly it will take on all of these values on the set of all y such that $y \geq \frac{1}{b}$. In fact, each value of $g(y)$ is repeated as many times as we like on this set.

What we have said implies that $f(x) = \sin \frac{1}{x}$ will take on all the values $-1 \leq f(x) \leq 1$ on the interval $[0,b]$, and, in fact, $f(x)$ oscillates between -1 and $+1$ an unlimited number of times in any interval $[0,b]$, no matter how small.

This is a statement both that f satisfies the conclusion of Theorem 8-2a, and that f is not continuous at $x = 0$.

For: Theorem 8-2a states that f, under certain conditions, will assume all values between 0 and $f(b)$ on $[0,b]$. We have seen that f in fact assumes all values between -1 and $+1$.

Continuity of f at 0 implies that for any $\epsilon > 0$, we can find a $\delta > 0$ such that for any x: $0 < |x| < \delta$, $|f(x) - f(0)| < \epsilon$. But we have seen that no matter how small we make δ, we can find values of x, $0 < |x| < \delta$, so that $f(x)$ takes on all values $-1 \leq f(x) \leq 1$, and $|f(x) - f(0)|$ takes on all values between 0 and 1 inclusive. We need only choose $\epsilon < 1$, and the condition for continuity is violated.



2. Prove that if f is continued and has an inverse on $[a,b]$ and $f(a) < f(b)$ then f is strictly increasing.

"f has an inverse f'" means that $f^{-1}(y_0)$ is uniquely **determined.** Formally, if $y_0 \neq y_1$, then $f(y_0) \neq f(y_1)$.

Suppose f is not strictly increasing that is, for some pair of $x_0 . x_1$ in $[a,b]$

$$x_0 < x_1,\ f(x_1) \leq f(x_0).$$

We will show that there is an x_2 in $[a,b]$ such that $x_2 \neq x_0$, $f(x_2) = f(x_0)$, which contradicts the assumption that f has an inverse. Consider the interval $[x_1,b]$.

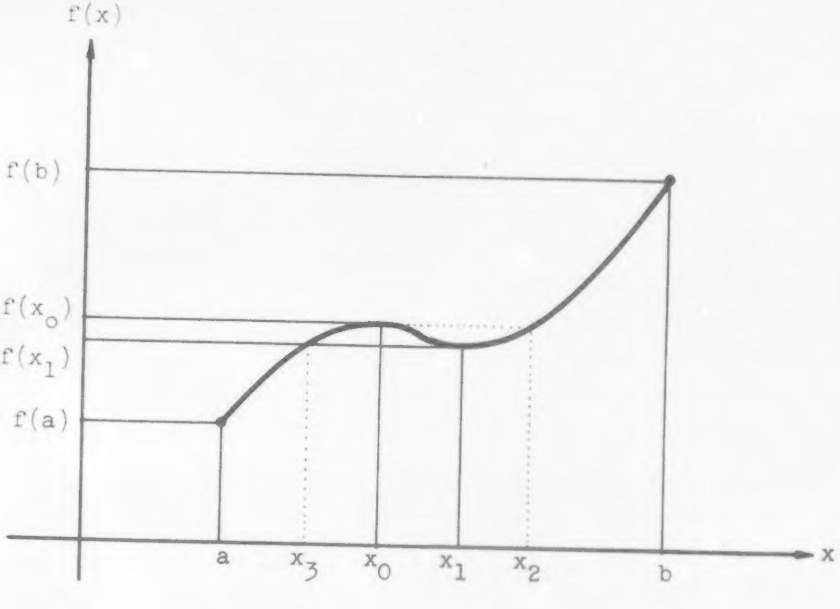
$$f(x_1) \leq f(x_0) = c \leq f(b).$$

By Theorem 8-2a, there is a value x_2 in $[x_1,b]$ such that

$$f(x_2) = c = f(x_0).$$

Certainly x_2 is in $[a,b]$, $x_2 > x_0$.

Similarly, we could show there is an x_3 in x_3 in $[a,x_0]$ such that $f(x_3) = f(x_1)$.

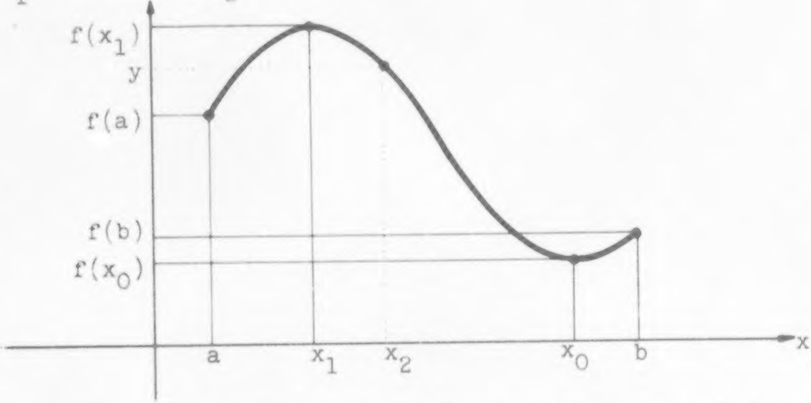


3. Prove that if f is continuous on $[a,b]$ then the image of $[a,b]$ is a closed interval.

From Theorem 8-2b, there are two points x_0 and x_1 with $a \leq x_0 \leq b$ and $a \leq x_1 \leq b$ such that

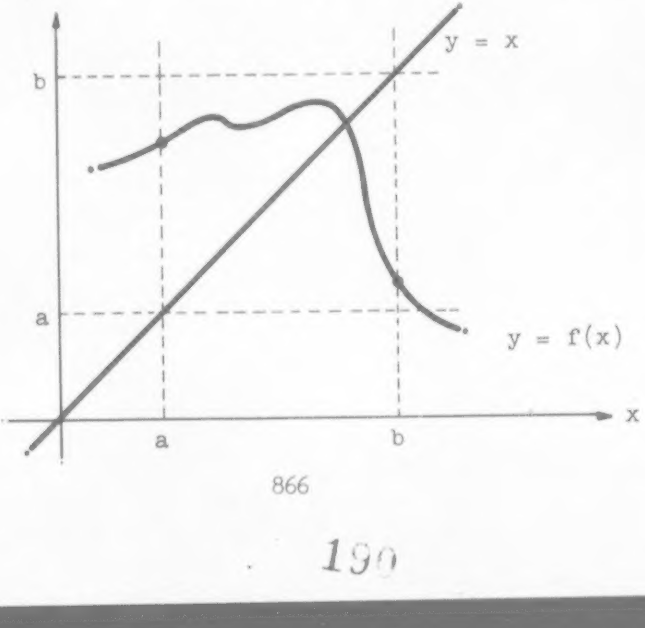
$$f(x_0) \leq f(x) \leq f(x_1) \quad \text{for all } x, \quad a \leq x \leq b.$$

The image of f, then, is clearly a subset of the closed interval $[f(x_0),f(x_1)]$ and includes the endpoints. Theorem 8-2a tells us that the image of f includes all the points in $[f(x_0,f(x_1)]$: (If we choose any value y, $f(x_0) < y < f(x_1)$, then there is a point x_2 between x_0 and x_1 such that $f(x_2) = y$.)

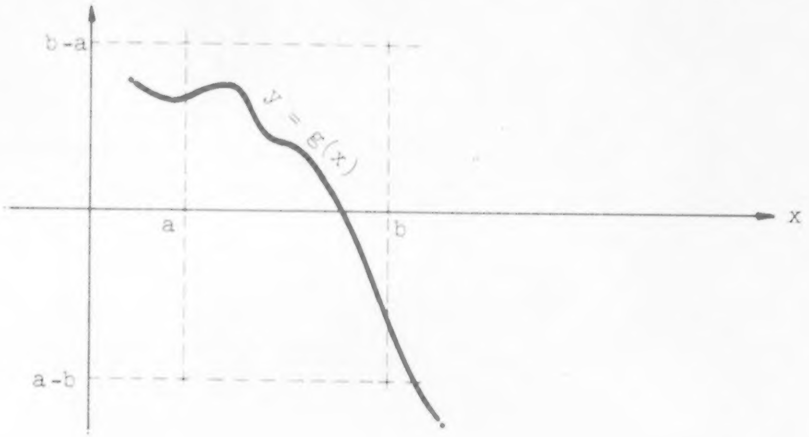


4. Prove that if f is continuous in $[a,b]$ and all values of f are in $[a,b]$ then there is an x on $[a,b]$ for which $f(x) = x$.

We could interpret this problem as proving that the graph of the function $f : x \to f(x)$ intersects the graph of the function $g : x \to x$



Note that this is analogous to the situation in Theorem 8-2a. In Theorem 8-2a, the line intersected was $y = d$, d between $f(a)$ and $f(b)$.



Consider the function

$$g(x) = f(x) - x.$$

This is clearly a continuous function, since it is the difference between two continuous functions. We wish to show that $g(x) = 0$ for some x in $[a,b]$.

(1) Suppose $g(x) > 0$ for all x in $[a,b]$.

Then $g(b) = f(b) - b > 0$, which is to say $f(b) > b$, which contradicts the assumption that the image of f is enclosed in $[a,b]$.

(2) Suppose $g(x) < 0$ for all x in $[a,b]$.

Similarly, this leads to the contradiction $f(a) < a$.

(3) Suppose $g(x)$ takes on both positive and negative values in $[a,b]$.

Choose c, d so that $g(c) < 0 < g(d)$ or $g(d) < 0 < g(c)$. Since g is continuous, Theorem 8-2a applies on $[c,d]$. Therefore we can find a point x_0 in $[c,d]$, such that $g(x_0) = 0$.

5. Suppose

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Does f satisfy the hypotheses of Theorem 8-2b on the interval $[0,1]$? Does (8) hold for f on $[0,1]$? on $[10^{-100},1]$?

f does not satisfy the hypotheses of Theorem 8-2b, since it is not continuous at $x = 0$. (See Problem 1.)

(8) does not hold on $[0,1]$. $f(x) = \frac{1}{x}$ can be made as large as we like if we choose x sufficiently close to $x = 0$.

(8) holds on $[10^{-100},1]$. lub $f(x) = 10^{100}$ on this interval.

6. Is the continuity of f essential to the hypothesis of (8)?

Yes. Consider the example of Exercise 5.

7. Can a discontinuous function whose domain is a closed interval be bounded?

Yes. Consider $x \to [2x]$ on $[0,1]$, or $x \to [nx]$.

8. Do Numbers 6 and 7 amount to the same question?

No. Number 6 asks, Is Theorem 8-2b true if we drop the hypothesis of continuity? while Number 7 asks, Is continuity necessary for boundedness?

9. Can a nonconstant function whose domain is the set of real numbers be bounded?

Yes; e.g., $x \to \sin x$ or $x \to \dfrac{x^2}{x^2+1}$.

10. Show that a function f which is increasing in some neighborhood of each point of an interval $[a,b]$ is an increasing function in $[a,b]$.

Consider the set A of points t in $[a,b]$ such that f is increasing in $[a,t]$. Call α the least upper bound of A. Then for $\beta > \alpha$, f is not increasing in (a,β).

We are given that f is increasing in a neighborhood of α if $\alpha < b$. Therefore, for some h, f is increasing in $(\alpha - h, \alpha + h)$. Hence, f is increasing in $(a, \alpha + h)$. But this means that $\alpha + h$ is in A, while α is an upper bound of A. So $\alpha \geq b$ and f is increasing in (a,b).

11. A function has the property that for each point of an interval where it is defined, there is a neighborhood in which the function is bounded. Show that the function is bounded over the whole interval. (This is an example where a local property implies a global one. It is clear that the global property here implies the local one.)

Let I be the interval for which f is locally bounded and let a and b be the respective left and right endpoints of I (no implication that I is either open or closed). Let A be the set of points consisting of the point a and those points α of I for which $f(x)$ is bounded on the interval $I_\alpha = I \quad \{x : x \leq \alpha\}$. Take $\bar{\alpha} = \sup A$. If $\alpha > \bar{\alpha}$, then f cannot be bounded on I_α. If $\bar{\alpha} = b$, then f is bounded on I. If $\bar{\alpha} < b$, then f is bounded on a neighborhood of $\bar{\alpha}$. It follows that f is bounded on the union of A and this neighborhood, contradicting that there is no interval I_α with $\alpha > \bar{\alpha}$ for which f is bounded.

Solutions Exercises A7-3

1. Prove Corollary 2 to Lemma A7-3.

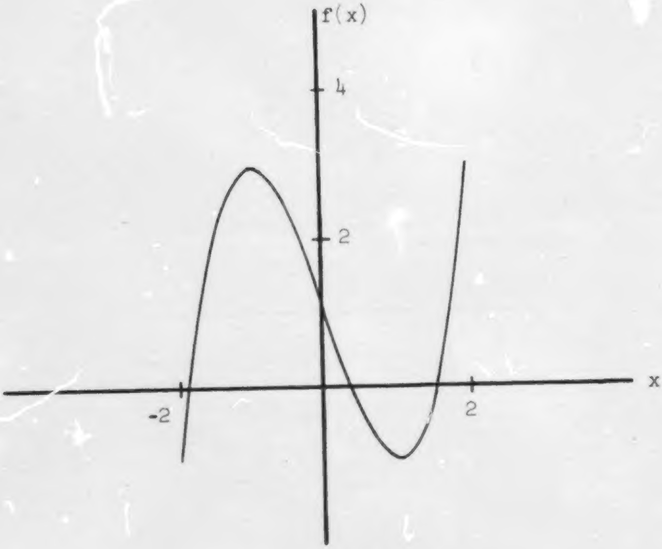
   Corollary 2. A polynomial of degree n can have no more than n distinct real roots.

   Let $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$, $a_n \neq 0$. The n-th derivative of p is given by $p^{(n)}(x) = n!a_n$, a nonzero constant function.

   It follows by Corollary 1 to Lemma A7-3 that $p^{(n-1)}$ has at most one real root. Applying this argument recursively we obtain the desired result.

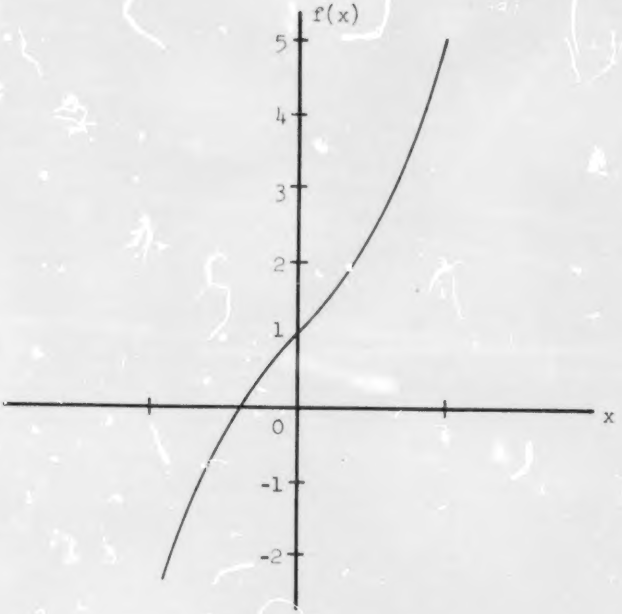
2. Sketch the graphs of the functions in Example A7-3a.

   (i)



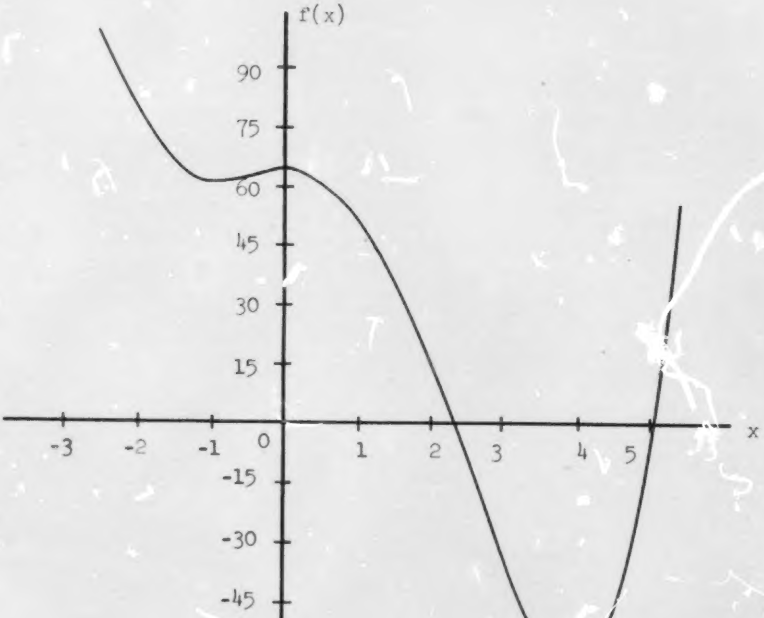
$f : x \rightarrow x^3 - 3x + 1$

(ii)



$f : x \longrightarrow x^3 + 3x + 1$

(iii)



$f : x \longrightarrow x^4 - 4x^3 - 8x^2 + 64$

3. Is the following converse of Rolle's Theorem true? If f is continuous on the closed interval $[p,q]$ and differentiable on the open interval (p,q), and if there is at least one point u in the open interval where $f'(u) = 0$, then there are two points m and n where $p \leq m < u < n \leq q$ such that $f(m) = f(n)$.

Not true. Counterexample: $y = x^3$ for any interval containing $x = 0$ in its interior.

4. Does Rolle's Theorem justify the conclusion that $\frac{dy}{dx} = 0$ for some values of x in the interval $-1 \leq x \leq 1$ for $(y + 1)^3 = x^2$?

If $(y + 1)^3 = x^2$, $\frac{dy}{dx} = \frac{2x}{3(y + 1)^2} = \frac{2}{3x^{1/3}}$. The conclusion of Rolle's Theorem does not hold for the closed interval $[-1,1]$ since $\frac{dy}{dx}$ does not exist at $x = 0$.

5. Given: $f(x) = x(x - 1)(x - 2)(x - 3)(x - 4)$. Determine how many solutions $f'(x) = 0$ has and find intervals including each of these without calculating $f'(x)$.

By Corollary 1 to Lemma A7-3, $f'(x) = 0$ has four solutions. There is one solution in each of the open intervals $(0,1)$, $(1,2)$, $(2,3)$, and $(3,4)$ since the zeros of f are $0, 1, 2, 3, 4$.

6. Verify that Rolle's Theorem (Lemma A7-3) holds for the given function in the given interval or give a reason why it does not.

(a) $f : x \rightarrow x^3 + 4x^2 - 7x - 10, \; [-1,2]$

(b) $f : x \rightarrow \frac{2 - x^2}{x^4}, \; [-1,1]$

(a)

$$f(x) = x^3 + 4x^2 - 7x - 10$$

$$f'(x) = 3x^2 + 8x - 7$$

$$f(-1) = f(2) = 0$$

One of the zeros of f', $\frac{-4 + \sqrt{37}}{3}$, is in the interval $(-1,2)$.

Thus, Rolle's Theorem holds.

(b) $f : x \to \frac{2 - x^2}{x^4}$

The function is not continuous at $x = 0$ and hence the theorem does not apply.

7. Prove that the equation

$$f(x) = x^n + px + q = 0$$

cannot have more than two real solutions for an even integer n nor more than three real solutions for an odd n. Use Rolle's Theorem.

$$f(x) = x^n + px + q$$

$$f'(x) = nx^{n-1} + p = 0$$

If n is even, f' is of odd degree and $f'(x) = 0$ has one and only one solution. It follows from Corollary 1 to Lemma A7-3 that f cannot have more than two real solutions in this case. Similarly, if n is odd, f' is of even degree and $f'(x) = 0$ has no more than two solutions. In this case $f(x) = 0$ has no more than three real solutions.

8. A function g has a continuous second derivative on the closed interval $[a,b]$. The equation $g(x) = 0$ has three different solutions in the open interval (a,b). Show that the equation $g''(x) = 0$ has at least one solution in the open interval (a,b).

By Corollary 1 to Lemma A7-3, g' has at least two zeros in the open interval (a,b) and hence the derivative of g', which is g'', must have at least one zero in the interval (a,b).

9. Show that the conclusion of the Mean Value Theorem does not follow for $f(x) = \tan x$ in the interval $1.5 < x < 1.6$ and explain why.

The theorem does not apply. The function f is not continuous on the open interval $(1.5,1.6)$. (Note that $1.5 < \frac{\pi}{2} < 1.6$ and $f(x)$ is not defined at $\frac{\pi}{2}$.) $\tan (1.5) > 0$, $\tan (1.6) < 0$. Therefore, if the Mean Value Theorem held in the interval $[1.5,1.6]$, there would exist a u in $(1.5,1.6)$ such that $f'(u) < 0$. But $D \tan x = \sec^2 x$, which is postive.

10. For each of the following functions show that the Mean Value Theorem fails to hold on the interval $[-a,a]$ if $a > 0$. Explain why the theorem fails.

(a) $f : x \to |x|$

$f(-a) = f(a) = a$. Yet $f'(x)$ is either $+1$ or -1 and never zero, so the Mean Value Theorem fails to hold. ($f'(0)$ does not exits.)

(b) $f : x \to \frac{1}{x}$

If the Mean Value Theorem holds for f on the interval $[-a,a]$, then for some u in $(-a,a)$, $f'(u) = \frac{4}{a^2}$. But $f'(x) = -\frac{1}{x^2}$ and so is never positive. (f is not continuous at $x = 0$.)

11. Show that the equation $x^5 + x^3 - x - 2 = 0$ has exactly one solution in the open interval $(1,2)$.

$f(x) = x^5 + x^3 - x - 2$ and $f'(x) = 5x^4 + 3x^2 - 1$. Since $f(1) < 0$ and $f(2) > 0$ the function f has a zero in the interval $(1,2)$, by the Intermediate Value Theorem and since $f'(x) > 0$ for $x > 1$, the function f has only one zero in the interval.

12. Show that $x^2 = x \sin x + \cos x$ for exactly two real values of x.

Let $f(x) = x^2 - x \sin x - \cos x$. Then

$$\begin{aligned} f'(x) &= 2x - x \cos x \\ &= x(2 - \cos x). \end{aligned}$$

$f'(x) = 0$ if and only if $x = 0$ and hence f has no more than two zeros. Since $f(0) = -1$ and $f(\pi) = f(-\pi) = \pi^2 + 1$ we conclude that there are zeros in the open intervals $(-\pi,0)$ and $(0,\pi)$, by the Intermediate Value Theorem.

13. Find a number that can be chosen as the number c in the Mean Value Theorem for the given function and interval.

(a) $f : x \to \cos x, \quad 0 \leq x \leq \frac{\pi}{2}$

$c = \arcsin\left(\frac{2}{\pi}\right)$

(b) $f : x \to x^3, \ -1 \leq x \leq 1$

$c = -\frac{\sqrt{3}}{3}$ or $\frac{\sqrt{3}}{3}$

(c) $f : x \to x^3 - 2x^2 + 1, \ -1 \leq x \leq 0$

$c = \frac{2 - \sqrt{13}}{3}$

(d) $f : x \to \cos x + \sin x, \ 0 \leq x \leq 2\pi$

$c = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

14. Derive each of the following inequalities by applying the Mean Value Theorem.

(a) $|\sin x - \sin y| \leq |x - y|$

If $f(x) = \sin x$, then f is continuous and differentiable for all x. By the Mean Value Theorem, for any $x \neq y$,

$$\frac{\sin x - \sin y}{x - y} = \cos u, \quad \text{where } x < u < y,$$

and hence

$$\left|\frac{\sin x - \sin y}{x - y}\right| = |\cos u| \leq 1,$$

or

$$|\sin x - \sin y| \leq |x - y|.$$

(b) $\frac{x}{1 + x^2} < \arctan x < x$ if $x > 0$.

If $g(x) = \arctan x$, g is continuous for $x \geq 0$ and differentiable for all $x > 0$. By the Mean Value Theorem,

$$\frac{\arctan x - \arctan 0}{x} = \frac{1}{1 + u^2}, \quad \text{where } 0 < u < x.$$

Since

$$\frac{1}{1 + x^2} < \frac{1}{1 + u^2} < 1,$$

we have

$$\frac{1}{1 + x^2} < \frac{\arctan x}{x} < 1,$$

so that

$$\frac{x}{1 + x^2} < \arctan x < x, \quad x > 0.$$

15. Use the Mean Value Theorem to approximate $\sqrt[3]{1.008}$.

Here $f(x) = \sqrt[3]{x}$ and we can choose $p = 1$, $q = 8$ for numerical simplicity. If we approximate $f(x)$ by the linear function g whose graph is the chord joining the points $(1,1)$ and $(8,2)$,

$$g(x) = 1 + \frac{1}{7}(x - 1).$$

Thus, an approximate value of $\sqrt[3]{1.008}$ is given by $g(1.008) = 1.001$. Since $|g(x) - f(x)| \leq 2M_1(x - p)$ where M_1 is an upper bound of $f'(x)$ in $[p,q]$, we have the error estimate

$$|1.001 - \sqrt[3]{1.008}| \leq 2(.008) \cdot \max \frac{1}{3x^{2/3}}$$

or

$$|1.001 - \sqrt[3]{1.008}| < 2(.008)(\tfrac{1}{3}) < .0054$$

and

$$0.9957 < \sqrt[3]{1.008} < 1.0064.$$

To get a better approximation, we choose a number q closer to 1. A convenient value is $q = \frac{27}{8}$. Then

$$g(x) = 1 + \frac{\frac{3}{2} - 1}{\frac{27}{8} - 1}(x - 1) = 1 + \frac{4}{19}(x - 1),$$

and

$$g(1.008) = 1 + \frac{4}{19}(.008) \approx 1.0017.$$

Here

$$|1.0017 - \sqrt[3]{1.008}| \leq 2(.008)\,\frac{1}{3}$$

and

$$.9963 < \sqrt[3]{1.008} < 1.0071.$$

Combining this with the previous approximation, we have

$$.9963 < \sqrt[3]{1.008} < 1.0064.$$

We can sharpen the upper bound by choosing q very close to 1.

Get arbitrarily close to $\sqrt[3]{1.008}$.

Alternatively, we can proceed as follows. From the Mean Value Theorem

$$\frac{f(q) - f(p)}{q - p} = f'(c), \qquad p < c < q,$$

it follows that

$$\frac{\sqrt[3]{1.008} - \sqrt[3]{1}}{1.008 - 1} = \frac{1}{3\sqrt[3]{c^2}}, \qquad 1 < c < 1.008,$$

(here $f(x) = \sqrt[3]{x}$, $q = 1.008$, $p = 1$). Then, since $f'(c)$ is monotonic in $[1, 1.008]$, we have

$$\frac{.008}{3\sqrt[3]{\frac{27}{8}}} < \frac{.008}{3\sqrt[3]{1.008^2}} < \sqrt[3]{1.008} - 1 < \frac{.008}{3}.$$

or

$$1.0017 < \sqrt[3]{1.008} < 1.0027.$$

16. Use the Mean Value Theorem to approximate $\cos 61^\circ$.

$$\frac{\cos q - \cos p}{q - p} = -\sin c, \qquad p < c < q.$$

Choose

$$p = \frac{\pi}{3}, \quad q = \frac{\pi}{3} + \frac{\pi}{180}.$$

Note that p and q are in radians and $1^\circ = \frac{\pi}{180}$ radians.

$$\frac{\cos\left(\frac{\pi}{3} + \frac{\pi}{180}\right) - \cos\frac{\pi}{3}}{\frac{\pi}{180}} = -\sin c.$$

Since

$$\sin 60^\circ < \sin c < 1,$$

$$-\frac{\pi}{180} \cdot 1 < \cos 61^\circ - \frac{1}{2} < -\frac{\pi}{180} \cdot \frac{1}{2}$$

or

$$\frac{1}{2} - \frac{\pi}{180} < \cos 61^\circ < \frac{1}{2} - \frac{\pi}{360}.$$

17. Show that $a\left(1 + \frac{\epsilon}{n(a^n + \epsilon)}\right) < \sqrt[n]{a^n + \epsilon} < a\left(1 + \frac{\epsilon}{na^n}\right)$ for $\epsilon > 0$, $a > 1$, $n > 1$ (n rational).

Let $f(x) = \sqrt[n]{x}$. Then

$$\frac{f(q) - f(p)}{q - p} = f'(u), \qquad p < u < q.$$

Choose $q = a^n + \epsilon$, $p = a^n$. Thus,

$$\frac{\sqrt[n]{a^n + \epsilon} - \sqrt[n]{a^n}}{\epsilon} = \frac{u^{1/n}}{n \cdot u}, \qquad a^n < u < a^n + \epsilon$$

and if $a > 1$,

$$\sqrt[n]{a^n + \epsilon} < \sqrt[n]{a^n + \epsilon} - a < \frac{\epsilon}{na^n} .$$

Since

$$\frac{\epsilon \sqrt[n]{a^n + \epsilon}}{n(a^n + \epsilon)} > \frac{\epsilon a}{n(a^n + \epsilon)} ,$$

we obtain the desired inequalities.

18. Using Number 17, obtain the following approximations:

(a) $3 + \frac{1}{10} < \sqrt[3]{30} < 3 + \frac{1}{9}$.

If $n = a = \epsilon = 3$ in the result of Number 17, then

$$3 + \frac{1}{10} < \sqrt[3]{30} < 3 + \frac{1}{9} .$$

(b) $3 + \frac{3}{5(244)} < \sqrt[5]{244} < 3 + \frac{1}{405}$.

If we set $n = 5$, $a = 3$, $\epsilon = 1$ in the result of Number 17, we obtain the desired result.

(c) Show that the approximation

$$\frac{1}{2}\left(3 + \frac{3}{5(244)} + 3 + \frac{1}{405}\right) \quad \text{to} \quad \sqrt[5]{244}$$

is correct to at least five decimal places.

We note that, in general, if $a < b < c$, then the error in taking $\frac{a + c}{2}$ as an approximation to b is no greater than $\frac{1}{2}(c - a)$. Letting $a = 3 + \frac{3}{5(244)}$, $b = \sqrt[5]{244}$, and $c = 3 + \frac{1}{405}$,

$$\frac{1}{2}\left(\frac{1}{405} - \frac{3}{5(244)}\right) = \frac{(1220 - 1215)}{2(405)(1220)} = \frac{1}{2(81)(1220)} < \frac{1}{100,000} .$$

Therefore,

$$\left|\frac{1}{2}\left(3 + \frac{1}{405} + 3 + \frac{3}{5(244)}\right)\right| = \sqrt[5]{244}| < 10^{-5}.$$

19. (a) Show that a straight line can intersect the graph of a polynomial function of n-th degree at most n times.

Let $y = mx + b$ be an equation of a straight line and $p(x) = a_0 + a_1x + \dots + a_nx^n$ be an n-th degree polynomial $(a_n \neq 0)$. Then if $y = p(x)$, we have $g(x) = 0$, where $g(x) = (a_0 - b) + (a_1 - m)x + a_2x^2 + \dots + a_nx^n$, $a_n \neq 0$. By Corollary 2 to Lemma A7-3 (proved in Solution No. 1), $g(x)$, a polynomial of n-th degree, has at most n distinct real roots. Therefore, a straight line can intersect the graph of a polynomial function of n-th degree at most n times (unless, of course, $g(x)$ is identically 0; i.e., p is linear and $p(x) = mx + b$).

(b) Obtain the corresponding result for rational functions.

Let $R(x) = \frac{f(x)}{g(x)}$ where f is of degree $t \geq 0$ and g of degree $s \geq 0$, $g(x) \neq 0$. When $R(x) = p(x)$, where p is a polynomial function of degree n, then $p(x) \cdot g(x) - f(x) = 0$. Let $q(x) = p(x) \cdot g(x) - f(x)$. Then the degree of q is at most $\max\{t, s + n\}$. Thus the graph of a rational function $x \to R(x) = \frac{f(x)}{g(x)}$, where f is of degree $t \geq 0$ and g is of degree $s \geq 0$, can intersect the graph of a polynomial of degree n in at most $\max\{t, s + n\}$ points. Since $n = 1$ for the linear function p, we conclude that the graph of a straight line can meet the graph of R in at most $\max\{t, s + 1\}$ points. (Again, if R is linear there is an exceptional case.)

(c) Could $\sin x$ or $\cos x$ be rational functions? Justify your answer.

No. By the results of (b), if $\sin x$ were a rational function, then the equation $\sin x = 0$ would have a finite number of solutions. The same remark applies to $\cos x$.

20. Prove the intermediate value property for derivatives; namely, if f is differentiable on the closed interval $[p,q]$ then $f'(x)$ takes on every value between $f'(p)$ and $f'(q)$ in the open interval (p,q).

Suppose $f'(p) < m < f'(q)$. Set $\epsilon = \min\{m - f'(p), f'(q) - m\}$. There exists a value δ satisfying $0 < \delta < q - p$ for which, simultaneously, $\left|\frac{f(p + \delta) - f(p)}{\delta} - f'(p)\right| < \epsilon$ and $\left|\frac{f(q) - f(q - \delta)}{\delta} - f'(q)\right| < \epsilon$.

For the function

$$g(x) = \frac{f(x + \delta) - f(x)}{\delta},$$

where δ is fixed and satisfies the preceding conditions, it follows that

$$g(p) < m < g(q - \delta).$$

The function $g(x)$ is continuous on the closed interval $[p, q - \delta]$ and therefore satisfies the intermediate value property on that interval. There must then exist a value r in $(p, q - \delta)$ such that

$$g(r) = \frac{f(r + \delta) - f(r)}{\delta} = m.$$

By the Mean Value Theorem we have for some value c, $f(r + \delta) - f(r) = \delta f'(c)$ where $r < c < r + \delta$. It follows that $f'(c) = m$.

Alternate Solution:

Let

$$r_p(x) = \frac{f(x) - f(p)}{x - p} \qquad \text{for } x \neq p,$$

$$r_p(x) = f'(p) \qquad \text{for } x = p.$$

r_p is continuous on $[p,q]$ and, by the Intermediate Value Theorem, takes on all values between $f'(p)$ and $\frac{f(q) - f(p)}{q - p}$. Similarly, let

$$r_q(x) = \frac{f(x) - f(q)}{x - q} \qquad \text{for } x \neq q,$$

$$r_q(x) = f'(q) \qquad \text{for } x = q.$$

r_q is continuous and takes on all values between $\frac{f(q) - f(p)}{q - p}$ and $f'(q)$ on $[p,q]$. Now by the Mean Value Theorem, there exists a c_p such that $f'(c_p) = r_p(x)$ and a c_q such that $f'(c_q) = r_q(x)$ for all x in $[p,q]$. Since $r_p(x)$ and $r_q(x)$ between them take on all values between $f'(p)$ and $f'(q)$, it follows by the Mean Value Theorem that $f'(x)$ takes on all values between $f'(p)$ and $f'(q)$.

21. Suppose

$$f'(x) \geq m > 0 \quad \text{and} \quad M \geq f''(x) > 0 \quad \text{on} \quad [a,b]$$

and that

$$f(r) = 0 \quad \text{where} \quad r \in [a,b].$$

Let $x_1 \in [a,b]$ and put

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} .$$

(a) Show that

$$|x_2 - r| \leq |x_1 - r|^2 \frac{M}{m} .$$

By the Mean Value Theorem, there is a ξ between x_1 and r such that

$$f'(\xi) = \frac{f(x_1) - f(r)}{x_1 - r}$$

and, since $f'(x)$ is also continuous and differentiable, there is a ξ_1 between x_1 and ξ such that

$$f''(\xi_1) = \frac{f'(x_1) - f'(\xi)}{x_1 - \xi}$$

$$f'(\xi) = f'(x_1) - (x_1 - \xi)f''(\xi_1).$$

Thus

$$\begin{aligned} f(x_1) - f(r) &= (x_1 - r)f'(\xi) \\ &= (x_1 - r)[f'(x_1) - (x_1 - \xi)f''(\xi_1)]. \end{aligned}$$

And $x_2 - r = x_1 - r - \dfrac{f(x_1) - f(r)}{f'(x_1)}$ since $f(r) = 0$

$$= x_1 - r - \frac{(x_1 - r)}{f'(x_1)}[f'(x_1) - (x_1 - \xi)f''(\xi_1)]$$

$$= (x_1 - r)(x_1 - \xi) \frac{f''(\xi_1)}{f'(x_1)} .$$

Since $\quad |x_1 - \xi| < |x_1 - r|$

$\quad |f''(\xi_1)| \leq M$

and $\quad |f'(x)| \geq m$

$$|x_2 - r| \leq |x_1 - r|^2 \frac{M}{m} .$$

(b) If $b - a < \frac{m}{M} k$, $0 < k < 1$, show that $|x_2 - r| \leq \frac{m}{M} k^2$.

$$|x_1 - r| < |b - a| < \frac{m}{M} k$$

from (a),

$$|x_2 - r| \leq (\frac{m}{M} k)^2 \frac{M}{m}$$

$$= \frac{m}{M} k^2.$$

Solutions Exercises A7-4

1. Let f be differentiable on a neighborhood of a point a for which $f'(a) = 0$. If $f'(x) \leq 0$ when $x < a$ and $f'(x) \geq 0$ when $x > a$ then $f(a)$ is a minimum. If $f'(x) \geq 0$ when $x < a$ and $f'(x) \leq 0$ when $x > a$ then $f(a)$ is a maximum. Give a proof.

   We consider the case for $f(a)$ a minimum. The proof for $f(a)$ a maximum is similar. Let x be a point of a deleted neighborhood of a. By the Mean Value Theorem there is a number u such that $f(x) - f(a) = f'(u)(x - a)$ for $a < u < x$ and for $x < u < a$. From the hypothesis, whether $x < a$ or $x > a$, it follows that

   $$f'(u)(x - a) \geq 0.$$

   We conclude that $f(x) \geq f(a)$ for all x in the neighborhood of a. Therefore $f(a)$ is a minimum.

2. Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). Suppose u is the one point in (a,b) where $f'(u) = 0$. Prove that if $f'(x)$ reverses sign in a neighborhood of u then $f(u)$ is the global extremum of f on $[a,b]$ appropriate to the sense of reversal.

   By the hypothesis, $f'(x) = 0$ has only one solution, $x = u$. The derivative $f'(x)$ must have constant sign in each of the intervals (a,u), (u,b) or we could find another zero of the derivative by Exercises A7-3, Number 17. Since $f'(x)$ reverses sign in a neighborhood of u, either $f'(x) > 0$ for $x < u$, or $f'(x) > 0$ for $x > u$. We will consider the case where $f'(x) > 0$ for $x < u$ and $f'(x) < 0$ for $x > u$ (the proof for the other case is similar). Since $f'(x)$ changes sign at u, $f(u)$ is a maximum in a neighborhood of u. By the Mean Value Theorem, for the interval $[a,u]$, $f(a) - f(u) = f'(v)(a - u)$ for some v, $a < v < u$. Thus, in this case $f'(v)(a - u) < 0$ and $f(a) < f(u)$. In the same way, applying the Mean Value Theorem to the closed interval $[u,b]$ we can show that $f(b) < f(u)$. The only extrema on $[a,b]$ are at the endpoints or at points where $f'(x) = 0$. Since we have eliminated the endpoints as possible maxima, we conclude that $f(u)$ is a global maximum of f on $[a,b]$.

3. Given a function f such that $f(1) = f(2) = 4$, and such that $f''(x)$ exists and is positive throughout the interval $1 \leq x \leq 3$. What can you conclude about $f'(2.5)$? about $f(2.5)$? Prove your statements, stating whatever theorems you use in your proof. (Note: This statement of the problem differs from that in the text.)

$f(1) = f(2) = 4$.

Since f'' exists on the interval $[1,3]$, f' is continuous and differential on $[1,3]$ and f also is continuous and differentiable on $[1,3]$. By Rolle's Theorem, there is a number u, $1 < u < 2$, such that $f'(u) = 0$. Since $f''(x) > 0$ on the interval $[1,3]$, f' is increasing on $[1,3]$ and hence, for $u < x < 3$, $f'(x) > f'(u) = 0$. Thus $f'(2.5) > 0$. Since $f'(x) > 0$ for x in $(u,3)$, f is increasing in $[u,3]$; since $u < 2$, it follows that $f(2.5) > f(2) = 4$.

4. Let f be a differentiable function on (a,b). Prove that the requirement that f be increasing is equivalent to the condition that $f'(x) \geq 0$ everywhere but that every interval contains points where $f'(x) > 0$.

Let us assume first that f increasing. If there were an entire interval on which $f'(x) = 0$ then by Corollary 1 to Theorem A7-4a it would follow that f is constant on that interval in contradiction to the assumption that f is strongly increasing. On the other hand, suppose that $f'(x) \geq 0$ but that every interval contains points where $f'(x) > 0$. Take any pair of points x_1, x_2 in (a,b) with $x_1 < x_2$. We will show $f(x_1) < f(x_2)$. By hypothesis, there is a point u in (x_1,x_2) where $f'(u) > 0$. Since $f'(u) = \lim\limits_{x \to u} \dfrac{f(x) - f(u)}{x - u}$ it follows by Lemma 3-4 that for some sufficiently small δ-neighborhood of u within (x_1,x_2)

$$\frac{f(x) - f(u)}{x - u} > 0. \tag{1}$$

Choose particular values p, q in this δ-neighborhood so that $p < u$ and $q > u$. We then have $x_1 < p < u < q < x_2$. It follows from (1) that

$$f(p) < f(u) < f(q). \tag{2}$$

However, under the assumption $f'(x) \geq 0$ in (a,b), we have from Theorem A7-4a that

$$f(x_1) \leq f(p) \quad \text{and} \quad f(q) \leq f(x_2). \tag{3}$$

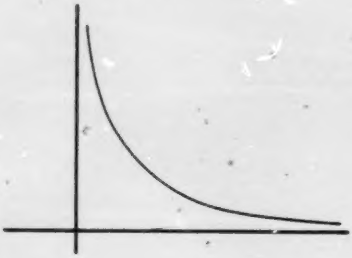
Combining the results of (2) and (3) we see that if $x_1 < x_2$ then $f(x_1) < f(x_2)$, i.e., that $f(x)$ is increasing.

5. A function g is such that g'' is continuous and positive in the interval (p,q). What is the maximum number of roots of each of the equations $g(x) = 0$ and $g'(x) = 0$ in (p,q)? Prove your result and give some illustrative examples.

We have g'' is continuous and positive on (p,q). Then g' is continuous and increasing on (p,q) and thus can have at most one real root (else $g'(x_1) = g'(x_2) = 0$ and g' is not increasing).

If $g'(x) = 0$ for any value of x, say x_0, then $g'(x) > 0$ for $x > x_0$ and $g'(x) < 0$ for $x < x_0$ (g' is increasing). Then g is increasing for $x > x_0$ and decreasing for $x < x_0$ and can have at most two real roots. If $g'(x) = 0$ for no value of x, then either $g'(x) < 0$ for all x or $g'(x) > 0$ for all x, and $g(x)$ is either increasing or decreasing for all x and can have at most one real root.

Case 1.



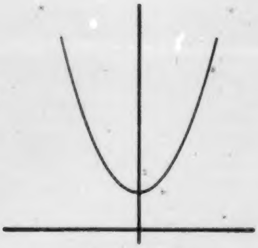
$g(x) = \frac{1}{x}$

$g''(x) > 0$

$g'(x)$ has no roots

$g(x)$ has no roots

Case 2.



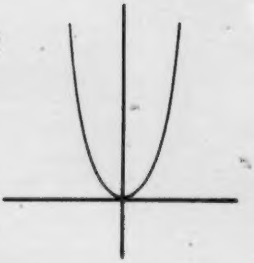
$g(x) = x^2 + 1$

$g''(x) > 0$

$g'(x)$ has no roots

$g(x)$ has no root

Case 3.



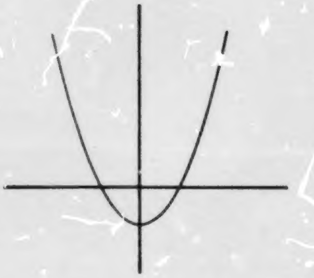
$g(x) = x^4 + x^2$

$g''(x) > 0$

$g'(x)$ has one root

$g(x)$ has one root

Case 4.



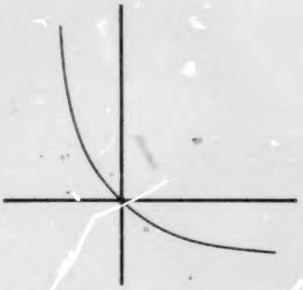
$g(x) = x^2 - 1$

$g''(x) > 0$

$g'(x)$ has one root

$g(x)$ has two roots

Case 5.



$g(x) = \frac{1}{x + 1} - 1$

$g''(x) > 0$

$g'(x)$ has no roots

$g(x)$ has one root

6. Suppose that $f'(a) = f''(a) = \ldots = f^{(n-1)}(a) = 0$ but that $f^{(n)}(a) \neq 0$. Determine whether $f(a)$ is a local extremum and if it is, which kind. (Hint: consider separately the cases n even and n odd.)

Let $f'(a) = f''(a) = \ldots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$, then

(i) whenever n is even and $f^{(n)}(a) > 0$, $f(a)$ is a local minimum value of f.

(ii) whenever n is even and $f^{(n)}(a) < 0$, $f(a)$ is a local maximum.

(iii) whenever n is odd, $f(a)$ is not a local extremum.

The proof of (i) is typical: From the proof of Theorem A7-4a and from Theorem A7-4c we know that $f^{(n-2)}(a)$ is an isolated minimum of $f^{(n-2)}$. In a deleted neighborhood of a we have $f^{(n-2)}(x) > f^{(n-2)}(a) \geq 0$. We conclude that the graph of $f^{(n-4)}$ is flexed upward in the neighborhood of a and hence that $f^{(n-4)}(a)$ is a minimum of $f^{(n-4)}$. Iterating this argument we obtain the desired result.

7. Let f be differentiable on an interval I. Prove that a necessary and sufficient condition that the graph of f be concave on an interval I is that the slope of the chord joining a point $(x,f(x))$ to a fixed point $(a,f(a))$ is a decreasing function of x. (It is understood that x and a lie in I.)

Note: The condition is required for every point a in I.

First we shall prove that if the graph of f is concave then the slope of chords through any fixed point a in I is a decreasing function.

Proof. Let a be any fixed point in I. Equations of chords through $(a,f(a))$ and $(q,f(q))$, for other points $q \neq a$ in I, will be of the form

$$y = g(x) = f(a) + (x - a)\frac{f(q) - f(a)}{q - a} .$$

The graph of f is concave only if $f(p) \geq g(p)$ for all p between a and q, i.e., for all p such that $a < p < q$ or $q < p < a$. In either case we have

$$f(p) \geq g(p) \geq f(a) + (x - a)\frac{f(q) - f(a)}{q - a} .$$

If $p > a$, then $q > p$, and this becomes

$$\frac{f(p) - f(a)}{p - a} \geq \frac{f(q) - f(a)}{q - a} .$$

If $p < a$, then $p > q$, and we have

$$\frac{f(p) - f(a)}{p - a} \leq \frac{f(q) - f(a)}{q - a} .$$

In both cases we have shown the slope function

$$x \to \frac{f(x) - f(a)}{x - a}$$

to be decreasing.

Now we shall prove the converse. We need to show that if the slope function for chords through any point a in I is decreasing, then if $y = g(x)$ is the (linear) equation of an arbitrary chord whose endpoints are $(b,f(b))$ and $(c,f(c))$, b and c in I, then $f(p) \geq g(p)$ for any p between b and c.

Proof. Let $b < p < c$. Then by hypothesis,

$$\frac{f(p) - f(b)}{p - b} \geq \frac{f(c) - f(b)}{c - b} .$$

The equation of the chord through $(b,f(b))$ and $(c,f(c))$ is

$$y = g(x) = f(b) + (x - b)\,\frac{f(c) - f(b)}{c - b}.$$

By definition of g,

$$\frac{g(x) - g(b)}{x - b} = \frac{f(c) - f(b)}{c - b},$$

so that

$$\frac{g(x) - f(b)}{x - b} \le \frac{f(p) - f(b)}{p - b},$$

for all x in I. In particular, taking $x = p$, we obtain

$$g(p) \le f(p).$$

8. (a) Let f be differentiable and its graph be concave on an interval I. Prove that the function

$$\phi(x) = \begin{cases} \dfrac{f(x) - f(a)}{x - a}, & x \neq a \\ f'(a), & x = a \end{cases}$$

is decreasing, where the fixed point a is any interior point of I.

We have already shown in Number 7 that ϕ is decreasing on I with the point a deleted. We now show that ϕ is decreasing on the entire interval.

Proof by Contradiction.

Suppose that $t > a$ and $\phi(a) < \phi(t)$.

Then for $\epsilon = \phi(t) - \phi(a)$ there exists a $\delta > 0$ such that

$$|\phi(x) - \phi(a)| < \epsilon \text{ whenever } |x - a| < \delta.$$

Choose a value x between a and t within the δ-neighborhood of a. We then have $x < t$, but

$$\phi(x) - \phi(a) \le |\phi(x) - \phi(a)| < \phi(t) - \phi(a),$$

whence $\phi(x) < \phi(t)$, in contradiction to the decreasing property of ϕ on the deleted interval.

A similar argument for $t < a$ completes the proof.

(b) From the result of (a), prove that a necessary and sufficient condition that the graph of f be concave on I is that f' be decreasing.

We first prove that if the graph of f is concave on I, then f' is decreasing on I.

Proof. Let a, b be points of I such that $a < b$. The function ϕ defined in part (a) and the function ψ defined by

$$\psi(x) = \begin{cases} \dfrac{f(x) - f(b)}{x - b} & , x \neq b \\ f'(b) & , x = b \end{cases}$$

are both decreasing on I. Since $\phi(b) = \psi(a)$, $\phi(a) \geq \phi(b)$, and $\psi(a) \geq \psi(b)$, it follows that

$$f'(a) \geq \phi(b) \geq \psi(a) \geq f'(b).$$

In other words, f' is decreasing on I.

Conversely, we show if f' is decreasing on I, then the graph of f is flexed downward on I.

Proof. For any points p, q, r of I with $p < q < r$ we have by the Mean Value Theorem

$$\frac{f(q) - f(p)}{q - p} = f'(t), \quad \text{where} \quad p < t < q$$

and

$$\frac{f(r) - f(q)}{r - q} = f'(u), \quad \text{where} \quad q < u < r.$$

Since $t < u$, and since f' is decreasing (by hyothesis) we have $f'(t) \geq f'(u)$. It follows that

$$\frac{f(q) - f(p)}{q - p} \geq \frac{f(r) - f(q)}{r - q} .$$

This inequality can be interpreted geometrically as a statement that the slope of the chord joining the fixed point $(q,f(q))$ to any point $(x,f(x))$ $x \in I$, is a decreasing function of x on I. It follows by Number 7 that the graph of f is concave.

9. (a) Let x and y be two points on an interval I in the domain of a function f. Show that a point is on the chord joining the points $(x,f(x))$ and $(y,f(y))$ on the graph of f if and only if its coordinates are

$$(\theta x + (1 - \theta)y, \theta f(x) + (1 - \theta)f(y))$$

for some θ such that $0 \leq \theta \leq 1$.

If $A(a',a)$ and $B(b',b)$ are points in a plane, then (x,y) is on the line $\overleftrightarrow{AB}$ if and only if

$$\frac{y - b}{x - a} = \frac{b' - b}{a' - a} \quad \text{or} \quad y = \frac{b' - b}{a' - a}(x - a) + b.$$

If we further require that (x,y) be on the segment $\overline{AB}$, then x must be between a and a', inclusive. Then $\frac{x - a}{a' - a}$ takes on all values between 0 and 1, inclusive. If we let $\theta = \frac{x - a}{a' - a}$

then $x = (a' - a)\theta + a$

and $y = (b' - b)\theta + b$

or $x = \theta a' + (1 - \theta)a,$

and $y = \theta b' + (1 - \theta)b,$

which was to be shown.

Since $\frac{x - a}{a' - a} = \theta$ takes on all values between 0 and 1 for $x : a < x < a'$ and $y : b < y < b'$, the converse is also true and all points $(\theta a + (1 - \theta)a', \theta b + (1 - \theta)b')$ are on $\overline{AP}$.

(b) Show that a differentiable function f is convex on I if and only if for all x and y in I and all $\theta : 0 \leq \theta \leq 1$,

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y).$$

Given a function f defined on an interval I, any point (x,y) on a chord of the graph of f will be of the form

$$x = \theta a + (1 - \theta)a',$$
$$y = \theta f(a) + (1 - \theta)f(a'),$$

where $\theta : 0 \leq \theta \leq 1$ and a', a are any two points on I. If we require that f is convex on I, then, by definition, all chords lie above their corresponding arcs, or

$$f(\theta a + (1 - \theta)a') \leq \theta f(a) + (1 - \theta)f(a').$$

This equation, then, is just the analytic characterization of f taken to be convex with graph flexed upward. Similarly, if the graph of f is flexed downward,

$$f(\theta a + (1 - \theta)a') \geq \theta f(a) + (1 - \theta)f(a').$$

(c) Use (b) to show that the graphs of the following functions are convex.

(i) $f : x \to ax + b.$

(ii) $f : x \to x^2$

(iii) $f : x \to -\sqrt{x}.$

(i) $f(x) = ax + b$

$$\begin{aligned} f(\theta x + (1 - \theta)y) &= a(\theta x + (1 - \theta)y) + b \\ &= \theta(ax + b) + (1 - \theta)(ay + b) \\ &= \theta f(x) + (1 - \theta)f(y). \end{aligned}$$

(ii) $f(x) = x^2$

$f(\theta x + (1 - \theta)y) = (\theta x + (1 - \theta)y)^2$

We want to show

$(\theta x + (1 - \theta)y)^2 \leq \theta x^2 + (1 - \theta)y^2.$

$$(\theta x + (1 - \theta)y)^2 - \theta x^2 - (1 - \theta)y^2.$$

$$= (\theta^2 - \theta)x^2 - 2(\theta^2 - \theta)xy + ((1 - \theta)^2 - (1 - \theta))y^2$$

$$= (\theta^2 - \theta)(x - y)^2 \leq 0$$

since $(x - y)^2 > 0$ and $\theta^2 - \theta \leq 0$ $(0 \leq \theta \leq 1)$.

(iii) $f(x) = -\sqrt{x}$

$f(\theta x + (1 - \theta)y) = \sqrt{\theta x + (1 - \theta)y}$

We want to show

$\sqrt{\theta x + (1 - \theta)y} \geq \theta\sqrt{x} + (1 - \theta)\sqrt{y}.$

If this is true, then

$$\theta x + (1 - \theta)y \geq \theta^2 x + 2\theta(1 - \theta)\sqrt{xy} + (1 - \theta)^2 y.$$

Since both sides are nonegative, then

$$(\theta - \theta^2)x - 2(\theta - \theta^2)\sqrt{xy} + (\theta - \theta^2)y \geq 0$$

or $(\theta - \theta^2)(\sqrt{x} - \sqrt{y})^2 \geq 0$, which is a true statement. Starting from this fact and reversing the steps taken above, we obtain the desired result.

10. (a) Derive the following property of differentiable functions. If the graph of f is concave on an interval I, then for all points a, b in I and any positive numbers p, q

$$f\left(\frac{pa + qb}{p + q}\right) \geq \frac{pf(a) + qf(b)}{p + q}.$$

In words, the function value of a weighted average is less than the weighted average of the function values.

We have for f that

$$f(\theta a + (1 - \theta)b) \geq \theta f(a) + (1 - \theta)f(b)$$

for all $\theta : 0 \leq \theta \leq 1$.

But $$0 \leq \frac{p}{p + q} \leq 1$$

and $$1 - \frac{p}{p + q} = \frac{q}{p + q}.$$

Setting $\theta = \frac{p}{p + q}$, we have

$$f(\frac{pa}{p + q} + \frac{qb}{p + q}) = f(\frac{pa + qb}{p + q}) \geq \frac{pf(a) + qf(b)}{p + q} .$$

(b) Prove that this property is sufficient for concavity.

Since $f(\frac{pa + qb}{p + q}) \geq \frac{pf(a) + qf(b)}{p + q}$ for all p, q positive, $\frac{p}{p + q}$ takes on all values between 0 and 1 exclusive. Then setting $\theta = \frac{p}{p + q}$, $f(\theta a + (1 - \theta)b) \geq \theta f(a) + (1 - \theta)f(b)$ for $\theta : 0 < \theta < 1$. For $\theta = 0$, $f(a) \geq f(a)$ and for $\theta = 1$, $f(b) \geq f(b)$. Then by Number 9, the graph of f is concave. Together (a) and (b) demonstrate necessity and sufficiency. The criterion then is an alternative way of characterizing differentiable functions.

11. Prove that if f is differentiable, then a necessary and sufficient condition for its graph to be concave is that

$$f(\frac{a + b}{2}) \geq \frac{f(a) + f(b)}{2} .$$

To prove necessity, use results established in Exercises A7-4, Number 10, and set $p = q = 1$.

To prove sufficiency we must show that if f is continuous and

$$f(\frac{a + b}{2}) \geq \frac{f(a) + f(b)}{2} \tag{1}$$

for all x, then the graph of f is concave. We observe from (1) that

$$\begin{aligned} f\left(\frac{x_1 + x_2 + x_3 + x_4}{4}\right) &\geq \frac{1}{2}\ f\left(\frac{x_1 + x_2}{2}\right) + f\left(\frac{x_3 + x_4}{2}\right) \\ &\geq \frac{1}{4}(f(x_1) + f(x_2) + f(x_3) + f(x_4)). \end{aligned}$$

Doubling the number of points repeatedly we obtain

$$f\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right) \geq \frac{1}{n}(f(x_1) + \dots + f(x_n)) \tag{2}$$

where n is a power of 2.

Now, if p and q are nonnegative integers with $p + q = n$ in (2) we obtain

$$f(\frac{pa + qb}{p + q}) \geq \frac{pf(a) + qf(b)}{p + q} . \tag{3}$$

To prove (3) for all real values of p and q, not just nonnegative values which add up to a power of 2, we must use the continuity of f. Convexity will then follow from the result of Exercise 10.

Setting $\theta = \frac{p}{p + q}$ we rewrite (3) in the form

$$f(\theta a + (1 - \theta)b) \geq \theta f(a) + (1 - \theta)f(b) \tag{4}$$

where $\theta = \frac{p}{p + q}$ is any rational number with denominator a power of 2 satisfying $o \leq \theta \leq 1$. For any value x_0 in $[a,b]$ we put

$$x_0 = ra + (1 - r)b$$

where r is now a real number satisfying $0 \leq r \leq 1$. Since f is continuous at x_0, for any positive ϵ we may choose a $\delta > 0$ so that $|x - x_0| < \delta$ insures $|f(x) - f(x_0)| < \epsilon$. We put $x = \theta a + (1 - \theta)b$ where θ is rational and has a power of 2 for its denominator. We have

$$|x - x_0| = |(\theta - r)(a - b)| = (b - a)|\theta - r| < \delta$$

provided $|\theta - r| < \frac{\delta}{b - a}$. It follows that $-\epsilon < f(x) - f(x_0) < \epsilon$. At the same time

$$\begin{aligned} &|\theta f(a) + (1 - \theta)f(b) - rf(a) - (1 - r)f(b)| \\ &\qquad = |\theta - r| \cdot |f(b) - f(a)| \\ &\qquad \frac{\delta|f(b) - f(a)|}{b - a}. \end{aligned}$$

We have

$$\begin{aligned} f(x_0) > f(x) - \epsilon &\geq \theta f(a) + (1 - \theta)f(b) - \epsilon \\ &\geq rf(a) + (1 - r)f(b) - \epsilon - \frac{\delta|f(b) - f(a)|}{b - a} \end{aligned}$$

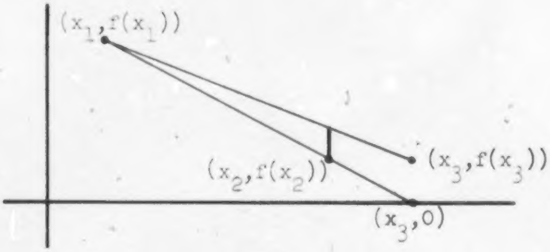
Given any $\epsilon^* > 0$, then, we take $\epsilon = \frac{\epsilon^*}{2}$ and $\delta^* \leq \delta$ but sufficiently small $\frac{\delta^*|f(b) - f(a)|}{b - a} < \frac{\epsilon^*}{2}$ and choose an approximation θ to r such that $|\theta - r| < \frac{\delta^*}{b - a}$. We then have $f(ra + (1 - r)b) \geq rf(a) + (1 - r)f(b) - \epsilon^*$ for each positive ϵ^*. This can only be true if

$$f(ra + (1 - r)b) \geq rf(a) + (1 - r)f(b).$$

We have extended Equation (4) to real values and the convexity of f follows.

12. The graph of a differentiable function f is concave and $f(x)$ is positive for all x. Show that f is a constant function.

Solution. Suppose f is nonconstant. Then there exist $x = x_1$ and $x = x_2$ such that $x_1 < x_2$ and $f(x_1) \neq f(x_2)$, say $f(x_1) > f(x_2)$. Now the line through $(x_1,f(x_1))$, $(x_2,f(x_2))$ intersects the x-axis at some point $x_3 > x_2$. We claim that $f(x_3) \leq 0$, which would contradict $f(x)$ positive. For if $f(x_3) > 0$, then the point $(x_2,f(x_2))$ would lie below the line segment connecting $(x_1,f(x_1))$ and $(x_3,f(x_3))$, contradicting f concave. So the assumption of f nonconstant has led to a contradiction.



13. Under what circumstances will the graph of a function f and its inverse both be concave? One concave and the other convex?

Without calculus: Let $[a,b]$ be an interval in the domain of f. If the graph of f on $[a,b]$ is concave, it is interior to the angle of $(a,f(a))$ formed by the ray to $(b,f(b))$ with the ray going vertically upward. Reflection in the line $y = x$ takes the upward ray into a ray going horizontally to the right and the other ray in the upper half-plane. If the angle was acute (f increasing) the reflected graph lies in the first quadrant below the chord. If the angle was obtuse (f decreasing) the reflected graph lies in the second quadrant above the chord. The same argument can be made algebraically.

With calculus: Assume f' and f'' exist. Then g' and g'' exist, where g is the inverse of f. We have

$$1 = [gf(x)]' = g'f(x) \cdot f'(x)$$

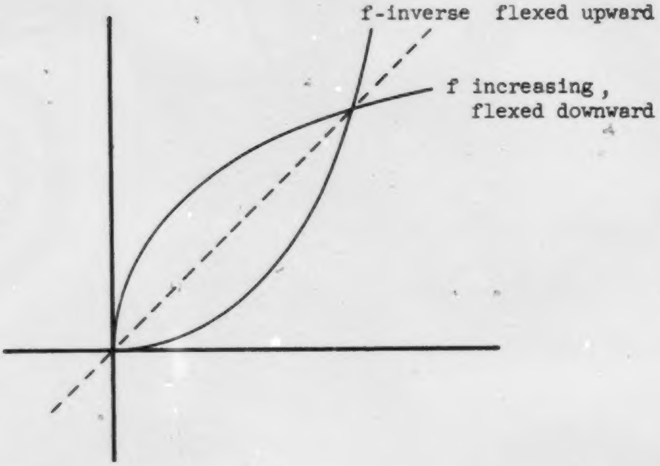
or

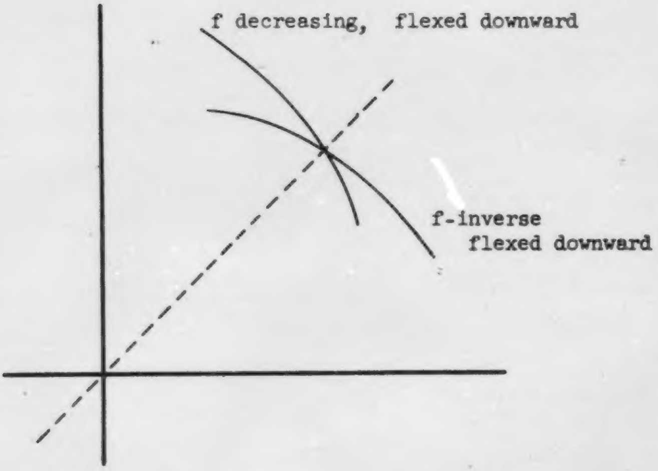
$$g'f(x) = \frac{1}{f'(x)}$$

$$g''f(x) \cdot f'(x) = \frac{-f''(x)}{[f'(x)]^2}$$

or $$g''f(x) = \frac{-f''(x)}{[f'(x)]^3} .$$

If $f''(x) < 0$, then $g''f(x) > 0$ if $f'(x) > 0$; $g''f(x) < 0$ if $f'(x) < 0$.


f-inverse flexed upward
f increasing,
flexed downward


f decreasing, flexed downward
f-inverse
flexed downward

14. If either of $D^2xF(x)$ or $D^2F(\frac{1}{x})$ is of one sign for $x > 0$, show that the other one has the same sign. Interpret geometrically and illustrate by several examples.

Define f and g by $f : x \to xF(x)$ and $g : x \to F(\frac{1}{x})$. We have

$$f''(x) = xF''(x) + 2F'(x)$$

and

$$g''(x) = \frac{F''(\frac{1}{x})}{x^4} + \frac{2F'(\frac{1}{x})}{x^3} = x^3f''(\tfrac{1}{x}).$$

Since $x^3 > 0$ for $x > 0$, the result follows immediately. Two routine examples follow.

(i)

$$F(x) = x^3 + x > 0$$

$$F'(x) = 3x^2 + 1 > 0$$

$$F''(x) = 6x > 0$$

$$F'(\tfrac{1}{x}) = \frac{3}{x^2} + 1 > 0$$

$$F''(\tfrac{1}{x}) = \frac{6}{x}.$$

$$xF''(x) + 2F'(x) > 0 \quad \text{for all} \quad x > 0$$

$$F''(\tfrac{1}{x})\left(\frac{1}{x^4}\right) + F'(\tfrac{1}{x})\,\frac{2}{x^3} > 0 \quad \text{for all} \quad x > 0.$$

(ii) $F(x) = \dfrac{x^4}{12} - \dfrac{x^3}{3} + x^2.$

$$F'(x) = \frac{x^3}{3} - x^2 + 2x; \quad F'(\tfrac{1}{x}) = \frac{1}{3x^3} - \frac{1}{x^2} + \frac{2}{x}$$

$$F''(x) = x^2 - 2x + 2; \quad f''(\tfrac{1}{x}) = \frac{1}{x^2} - \frac{2}{x} + 2.$$

$$xF''(x) + 2F'(x) = x^3 - 2x^2 + 2x + \frac{2x^3}{3} - 2x^2 + 4x$$

$$= x(\frac{5x^2}{3} - 4x + 6) > 0. \quad \text{(Discriminant} < 0)$$

$$\frac{F''(\frac{1}{x})}{x^4} + \frac{2F'(\frac{1}{x})}{x^3} = \frac{1}{x^6} - \frac{2}{x^5} + \frac{2}{x^4} + \frac{2}{3x^6} - \frac{2}{x^5} + \frac{4}{x^4}$$

$$= \frac{1}{x^6}(\frac{5}{3} - 4x + 6x^2) > 0.$$

If the graph of $x \to xF(x)$ is convex (or concave), then the graph of $x \to F(\frac{1}{x})$ is convex (or concave).

15. If the graph of f is concave and $F(a) = F(b) = F(c)$ where $a < b < c$, show that $F(x)$ is constant in (a,c).

If f' were not constant, then since the curve lies nowhere below its chord we have $f(u) < f(a)$. But $(b,f(b))$ then lies below both the chords from $(a,f(a))$ to $(u,f(u))$ and from $(u,f(u))$ to $(b,f(b))$, contradicting concavity.

16. (a) Let a, b, c be three points in I such that $a < b < c$, and suppose that the graph of f is convex in I. Show that

$$f(b) \leq \frac{c-b}{c-a} f(a) + \frac{b-a}{c-a} f(c).$$

(Hint: use the result of Number 9).
Hence, show that

$$f(a) \geq \frac{c-a}{c-b} f(b) - \frac{b-a}{c-b} f(c),$$

$$f(c) \geq \frac{c-a}{b-a} f(b) - \frac{c-b}{b-a} f(a).$$

Similarly to 4(a) (this is less complicated as we are on the number line here rather than the plane), $b = \theta a + (1-\theta)c$, for some $\theta : 0 < \theta < 1$. Then

$$\theta \cdot (a - c) = b - c$$

or

$$\theta = \frac{c-b}{c-a}, \ (1-\theta) = \frac{b-a}{c-a}.$$

and

$$b = \frac{c-b}{c-a} a + \frac{b-a}{c-a} c.$$

Now from 9(b), we have that

$$f(b) \leq \frac{c-b}{c-a} f(a) + \frac{b-a}{c-a} f(a),$$

for the graph of f flexed upward. Multiplying both sides by $\frac{c-a}{c-b}$, we get $f(a) > f(b) \cdot \frac{c-a}{c-b} - f(c) \cdot \frac{b-a}{c-b}$. Similarly $f(c) \geq f(b) \cdot \frac{c-a}{b-a} - f(a) \cdot \frac{c-b}{b-a}$.

(b) If the graph of F is convex in a closed interval, show that F is bounded in the interval.

It is an immediate consequence of the flexure of f that it is bounded above. Given a closed interval $[\alpha,\beta]$ in I, let the chord connecting $(\alpha,f(\alpha))$, $(\beta,f(\beta))$ be described by the linear function g. Then $g(x) \leq \max\{f(\alpha),f(\beta)\}$ on $[\alpha,\beta]$, and by the upward flexure of f, $f(x) \leq g(x) \leq \max\{f(\alpha),f(\beta)\}$.

To find lower bounds for f, we use part (a). Take any point γ in $[\alpha,\beta]$ which is not an endpoint, so that $\alpha < \gamma < \beta$. For $\alpha \leq x < \gamma$, we have $x < \gamma < \beta$ and

$$\begin{aligned} f(x) &\geq \frac{\beta - x}{\beta - \gamma} f(\gamma) - \frac{\gamma - x}{\beta - \gamma} f(\beta) \qquad \text{by (a)}, \\ &\geq -\left(\frac{\beta - x}{\beta - \gamma} |f(\gamma)| - \frac{\gamma - x}{\beta - \gamma} |f(\beta)|\right) \\ &\geq -\left(\frac{\beta - \alpha}{\gamma - \alpha} |f(\gamma)|\right). \end{aligned}$$

Similarly, for $\gamma < x \leq \beta$, we have $\alpha < \gamma < x$ and using (a) again,

$$f(x) \geq -\left(\frac{\beta - \alpha}{\gamma - \alpha} |f(\gamma)|\right).$$

(c) Show by a counter example that the result in (a) is not valid for an open interval.

The graph of $f : x \to \frac{1}{x}$ is convex on $(0,x_0)$, $x_0 > 0$, but is unbounded. However, f is always bounded below on a finite open interval as well as finite closed interval, as can be seen from the proof of (b).

Appendix 8

MORE ABOUT INTEGRALS

Solutions Exercises A8-1

1. Let f be a function which takes on a maximum and minimum on every closed interval (e.g., f could be a continuous function, or monotone). Let $U^*(\sigma)$ and $L^*(\sigma)$ be the upper and lower Riemann sums obtained by using the maximum and minimum values of $f(x)$ as the appropriate bounds in each interval of the subdivision.

   Let σ_1 and σ_2 be any partitions of $[a,b]$. Prove for the joint subdivision $\sigma = \sigma_1 \cup \sigma_2$ that

$$U^*(\sigma_1) \geq U^*(\sigma) \geq L^*(\sigma) \geq L^*(\sigma_2).$$

   In other terms, by adding new points to a subdivision we may reduce the difference between the upper and lower Riemann sums, and we cannot increase it.

   Let $\sigma_1 = \{x_0, x_1, \ldots, x_n\}$ and consider the partition τ of the subinterval, $\tau = \{u_0, u_1, \ldots, u_p\}$ where $u_0 = x_{k-1}$ and $u_p = x_k$. and $u_1, u_2, \ldots, u_{p-1}$ are those points (if any) of σ_2 which lie in the interior of $[x_{k-1}, x_k]$. If M_k is the maximum of $f(x)$ in $[x_{k-1}, x_k]$, then for the maximum r_j of $f(x)$ in any subinterval $[u_{j-1}, u_j]$ we have $r_j \leq M_k$. It follows that

$$M_k(x_k - x_{k-1}) \geq \sum_{j=1}^{p} r_j(u_j - u_{j-1}).$$

   Thus we have compared the k-th term in $U^*(\sigma_1)$ with the sum of those terms in $U^*(\sigma)$ which correspond to the subintervals of $[x_{k-1}, x_k]$. It follows on addition that

$$U^*(\sigma_1) \geq U^*(\sigma).$$

   In the same way, show that

$$L^*(\sigma_2) \leq L^*(\sigma).$$

   Since $U^*(\sigma)$ and $L^*(\sigma)$ are upper and lower sums for the same partition we have

$$L^*(\sigma) \leq U^*(\sigma)$$

   from which the result now follows.

2. Consider the function f defined on $[0,1]$ by

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1, & x \text{ rational.} \end{cases}$$

Prove that the integral of f does not exist.

Every interval contains both rational and irrational points. Consequently the maximum of $f(x)$ in every interval is 1 and the minimum is 0.. For any partition σ of $[0,1]$ we have for any upper sum U and any lower sum L,

$$U - L = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1})$$

where $M_k \geq 1$ and $m_k \leq 0$. It follows that

$$M_k - m_k \geq 1$$

and

$$U - L \geq \sum_{k=1}^{n} (x_k - x_{k-1})$$

$$\geq 1$$

for all upper and lower sums.

The criterion of Theorem 6-3a cannot be satisfied and f is not integrable over $[0,1]$.

3. Consider the function f defined on $[0,1]$ by

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ \frac{1}{t}, & x \text{ rational, } x = \frac{s}{t} \text{ in lowest terms.} \end{cases}$$

Prove that the integral of f over $[0,1]$ exists and find its value.

As an upper bound on the number of rational points with a given denominator $t > 1$ and numerator s relatively prime to t and less than t we have $t - 1$. For a fixed positive integer q introduce the partition

$$\sigma = \{\frac{1}{2q^2}, 1 - \frac{1}{2q^2}, \frac{s}{t} \pm \frac{1}{(t-1)2q^2}, (t = 2, \ldots, q)\}.$$

Thus each point of the form $\frac{s}{t}$, $t > 1$, is contained in a subinterval of the partition of length at most $\frac{1}{(t-1)q^2}$.

Since $f(\frac{s}{t}) = \frac{1}{t} < 1$,, we see that the maximum contribution to the upper sum of the rational points with denominator t, $(t \geq 1)$ is $\frac{1}{q^2}$. Taking the sum of the contributions of all rational points with denominators $t = 1, 2, \ldots, q$, we obtain a contribution no greater than $\frac{q}{q^2} = \frac{1}{q}$. In the remaining intervals of the subdivision we have $f(x) < \frac{1}{q}$ and since the total length of the remaining subintervals is at most 1, we have a contribution to the upper sums of no more than $\frac{1}{q}$. In this way we have found an upper sum U over σ for which $U < \frac{2}{q}$. Since we can always take the lower sum $L = 0$ and q may be any positive integer whatever, it follows that the integral exists and has the value 0.

4. Give an example of a nonintegrable function fg where f and g are each integrable.

Let g be the function defined in Number 5 and take $f = \text{sgn}$. The function fg is then the given function of Number 4.

TC A8-2. The Integral of a Continuous Function

The method of establishing the existence of the integral of a continuous function given here is quite different than that given in SMSG Calculus, pp. 645ff. Our proof does not make use of the fact that a continuous function on a closed interval is uniformly continuous. By using properties of least upper and greatest lower bounds, and Theorem 8-2b we show directly that the Area Theorem holds for upper integrals, and lower integrals, then use Theorem 7-3b to establish that these are equal. Other discussions of this same technique can be found in the calculus books of Begle, Lang and Richmond.

Solutions Exercises A8-2

1. Show that if $x \in [a,b]$ and $\delta > 0$ then $[a,b] \quad [x - \delta, x + \delta]$ is a closed interval. (Hint: Let a_1 be the larger of a and $x - \delta$, b_1, the smaller of b and $x + \delta$ and show that $[a_1,b_1] = [a,b] \quad [x - \delta, x + \delta]$).

If $t \geq a_1$ then since $a_1 \geq a$ and $a_1 \geq x - \delta$ we certainly have

$$t \geq a \text{ and } t \geq x - \delta.$$

Likewise, if $t \leq b_1$, then

$$t \leq b \text{ and } t \leq x + \delta.$$

Thus $a_1 \leq t \leq b_1$ implies $t \in [a,b] \cap [x - \delta, x + \delta]$.

Conversely, if $t \in [a,b] \cap [x - \delta, x + \delta]$ then $a \leq t \leq b$ and $x - \delta \leq t \leq x + \delta$ so that

$$\max(a, x - \delta) \leq t \leq \min(b, x + \delta)$$

and hence $t \in [a_1,b_1]$. (See Figure 1.)

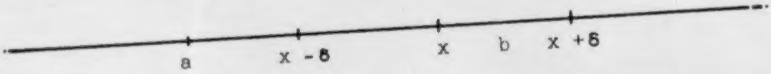


Figure 1

2. Show that if $x' > x$ and $x' \epsilon [a,b]$, $x \epsilon [a,b]$ then $[x,x']$ is a subinterval of $[a,b]$.

This is quite simple, for if $x \leq t \leq x'$ then, since $a \leq x \leq b$ and $a \leq x' \leq b$, we certainly have $a \leq t \leq b$. (See Figure 2.)

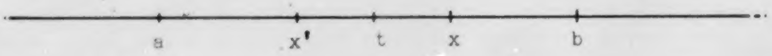


Figure 2

3. Show that

$$\int_{\underline{a}}^{b} f = -\int_{a}^{\overline{b}} (-f).$$

Suppose $\sigma = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$ and that

$$m_k \leq f(x) \quad \text{for} \quad x \epsilon [x_k, x_{k+1}].$$

Then

$$L = \sum_{k=1}^{n} m_k(x_k - x_k - 1)$$

is a lower sum for f, while, since $-f(x) \leq -m_k$ on $[x_k, x_{k+1}]$,

$$U = \sum_{k=1}^{n} (-m_k)(x_k - x_{k-1})$$

is an upper sum for $-f$. Since $\int_a^b (-f)$ is the greatest lower bound of all upper sums we must have

$$\int_a^{\overline{b}} (-f) \leq U.$$

That is,

$$-U \leq -\int_a^{\overline{b}} (-f).$$

Since $-U = L$, this gives

$$L \leq -\int_a^{\overline{b}} (-f)$$

so that $-\int_a^{\overline{b}} (-f)$ is an upper bound for the lower sums of f, and hence cannot be less than the least $\int_{\underline{a}}^{b} f$ of the lower sums, that is,

$$\int_{\underline{a}}^{b} f \le \int_a^{\overline{b}} (-f).$$

A similar argument establishes the reverse inequality.

4. Deduce from Number 3 and Theorem A8-2 that $\underline{F}' = f$ if f is continuous on $[a,b]$.

Let $\overline{G}(x) = \int_a^{\overline{b}} (-f)$ and $\underline{F}(x) = \int_{\underline{a}}^{b} f$. Number 3 gives

$$\underline{F}(x) = -\overline{G}(x)$$

while Theorem A8-2 gives $\overline{G}' = -f$. Hence

$$D\underline{F} = D(-\overline{G}) = -D(\overline{G}) = -(-f) = f.$$

5. Show that if f is continuous on $[a,b]$, then there is a number c in $[a,b]$ such that

$$\int_a^b f = (b - a)f(c).$$

(Hint: Choose c_1 and d_1 in $[a,b]$ such that $f(c_1)$ and $f(d_1)$ are the respective maximum and minimum of f on $[a,b]$. Show that

$$f(d_1) \le \frac{\int_a^b f}{b - a} \le f(c_1)$$

and apply the Intermediate Value Theorem.

Lemma A8-2b gives

$$f(d_1)(b - a) \le \int_a^{\overline{b}} f \le f(c_1)(b - a)$$

Since f is continuous, $\int_a^b f = \int_a^{\overline{b}} f$, so that

$$f(d_1) \le \frac{\int_a^b f}{b - a} \le f(c_1)$$

that is

$$\frac{\int_a^b f}{b - a} = d$$

lies between two values of the continuous function f. The Intermediate Value Theorem then gives a number d between c_1 and d_1 such that $f(c) = d$.

6. Use the Mean Value Theorem to show that Number 5 is true. Can you then choose c so that $a < c \leq b$?

Let $F(x) = \int_a^x f$, so that

$$F(a) = 0 \text{ and } F'(x) = f(x).$$

The Mean Value Theorem gives the existence of c, $a < c < b$ so that

$$\frac{F(b) - F(a)}{b - a} = F'(c)$$

that is

$$\frac{1}{b - a}\int_a^b f = f(c).$$

Thus c can be chosen in the open interval (a,b).

7. Show that if f is continuous and nonnegative on $[a,b]$ with $a < b$ and if $f(x) > 0$ for some x in $[a,b]$ $\int_a^b f > 0$. (Hint: Show that there is a $\delta > 0$ and $m > 0$ such that $f(x) \geq m$ on $[a,b] \cap [x - \delta, x + \delta]$.)

Since f is continuous at x, for $\epsilon = \frac{f(x)}{2}$ we can find $\delta_1 > 0$ such that

$$|f(t) - f(x)| < \epsilon$$

if $t \in [a,b] \cap (x - \delta_1, x + \delta_1)$, $t \neq x$, (i.e., t is in the domain of f and $0 < |t - x| < \delta_1$). Thus

$$-\epsilon < f(t) - f(x) < \epsilon$$

so, in particular

$$f(t) > f(x) - \epsilon = \frac{f(x)}{2}.$$

Take $m = \frac{f(x)}{2}$, δ any positive number δ_1. Then $t \in [a,b] \cap [x - \delta, x + \delta]$ implies

$$f(t) \geq m.$$

Now let $[a_1, b_1] = [a,b] \cap [x - \delta, x + \delta]$ and write

$$\int_a^b f = \int_a^{a_1} f + \int_{a_1}^{b_1} f + \int_{b_1}^{b} f.$$

Note that

$$\int_a^{a_1} f \geq 0, \quad \int_{b_1}^{b} f \geq 0 \qquad (\text{since } f \geq 0),$$

and

$$\int_{a_1}^{b_1} f \geq m(b_1 - a_1) > 0 \qquad (\text{since } a_1 < b_1),$$

so that

$$\int_a^b f \geq m(b_1 - a_1) > 0.$$

8. Deduce from Number 7 that if $f'(x) > 0$ for $a < x < b$ and f' is continuous on $[a,b]$ then f is strictly increasing on $[a,b]$.

If $a \leq x_1 < x_2 \leq b$ then

$$\int_{x_1}^{x_2} f' = f(x_2) - f(x_1)$$

which must be positive from Number 7, that is $f(x_2) > f(x_1)$.

9. Suppose

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 2, & 1 < x \leq 2 \end{cases}$$

(a) Show directly from the definition and properties of upper integrals that:

$$\overline{F}(x) = \overline{\int_0^x} f = \begin{cases} x & , 0 \leq x \leq 1 \\ 2x - 1, & 1 < x \leq 2 \end{cases}$$

For $0 \leq x \leq 1$, we can choose the subdivision $\sigma = \{0,x\}$ of $[0,x]$ and the upper and lower sums

$$U = 1(x - 0)$$
$$L = 1(x - 0)$$

to obtain

$$x = L \leq \int_0^{\overline{x}} f \leq U = x$$

so that

$$\overline{F}(x) = \int_0^{\overline{x}} f = x \qquad \text{for } 0 \leq x \leq 1.$$

For $1 < x \leq 2$, we can choose the subdivision of $[0,x]$.

$J_n = \{0, 1 - \frac{1}{n}, 1 + \frac{1}{n}, x\}$ where n is any positive integer such that $1 + \frac{1}{n} < x$.

Let

$$U_n = 1[(1 - \tfrac{1}{n}) - 0] + 2[(1 + \tfrac{1}{n}) - (1 - \tfrac{1}{n})] + 2[x - (1 + \tfrac{1}{n})]$$

$$L_n = 1[(1 - \tfrac{1}{n}) - 0] + 1[(1 + \tfrac{1}{n}) - (1 - \tfrac{1}{n})] + 2[x - (1 + \tfrac{1}{n})]$$

These are then upper and lower sums for J_n, so that

$$L_n \leq \int_0^{\overline{x}} f \leq U_n.$$

Since

$$U_n = 2x - 1 + \frac{1}{n}$$

$$L_n = 2x - 1 - \frac{1}{n}$$

we have

$$2x - 1 - \frac{1}{n} \leq \int_0^{\overline{x}} f \leq 2x - 1 + \frac{1}{n}.$$

This holds for all n sufficiently large so we conclude that

$$\overline{F}(x) = \int_0^{\overline{x}} f = 2x - 1 \quad \text{for } 1 < x \leq 2.$$

(b) Does $\overline{F}$ have a derivative at $x = 1$?
Why doesn't this contradict Theorem A8-2?

$\overline{F}$ doesn't have a derivative at $x = 1$, for if $h > 0$ then $1 + h > 1$ and

$$\frac{\overline{F}(1 + h) - \overline{F}(1)}{h} = \frac{(1 + h) - 1}{h} = 1$$

so the right hand and left hand limits are not the same.

This doesn't contradict Theorem A8-2 as f is not continuous at $x = 1$.

10. Suppose f is bounded on $[a,b]$ and $\overline{F}(x) = \int_a^{\overline{x}} f$. Show that $\overline{F}$ is continuous on $[a,b]$. (Hint: Make use of Lemmas A8-2a, b, which hold for bounded functions.)

If $|f(x)| \leq M$, $a \leq x \leq b$, then

$$-M \leq f(x) \leq M, \ a \leq x \leq b.$$

Therefore, if $x_1 < x_2$, apply Lemmas A8-2a and b to obtain

$$\overline{F}(x_2) - \overline{F}(x_1) = \int_a^{\overline{x}_2} f - \int_a^{\overline{x}_1} f = \int_{x_1}^{\overline{x}_2} f$$

so that

$$-M(x_2 - x_1) \leq \overline{F}(x_2) - \overline{F}(x_1) \leq M(x_2 - x_1).$$

Therefore, as $x_2 \to x_1$, $x_2 > x_1$, we have

$$\overline{F}(x_2) - \overline{F}(x_1) \to 0.$$

A similar result holds for $x_2 < x_1$ and establishes that

$$\lim_{x_2 \to x_1} \overline{F}(x_2) = \overline{F}(x_1).$$